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An Infinity Differentiable Weight Function for Smoothed Article Hydrodynamics Approaches

DaMing Yuan[1], Hao Fu[2], JianJie Yu[3]
1,2 College Of Mathmatics and Information Science, Nanchang Hangkong University,Nanchang,330063,P R China
3 College Of Civil Engineer and Architecture, Nanchang Hangkong University,Nanchang,330063,P R China

Abstract:- In this paper, we make a brief review of smoothed particle hydrodynamic method and develop a new infinity differentiable weight function in the three-dimensional space. The function satisfies the demanded properties. Furthermore, we investigate the consistency and analyze the estimation errors of the SPH interpolation with the weight function.

Keywords:- weight function, smoothed particle hydrodynamic, meshless Method

I. INTRODUCTION

In the past thirty years, meshless methods have been developed excellently. In these methods, the central and most important issue is how to create shape functions using only nodes scattered arbitrarily in a domain without any predefined mesh. Development of more effective methods for constructing shape functions is thus one of the hottest area of research in the area of meshless methods. A number of ways to construct shape functions have been proposed and these methods can be classified into three major categories [1]: (1) Finite integral representation methods; (2) Finite series representation methods; (3) Finite differential representation methods. Among these methods, the finite integral representation methods are relatively young, but have found a special place in Meshless methods with the development of smoothed particle hydrodynamics

The term smoothed particle hydrodynamics (SPH) was coined by Gin-gold and Monaghan in 1977 [2]. By using a kernel estimation technique to estimate probability densities from sample values, Gingold and Monaghan[2] and Lucy [3] independently proposed a way to reproduce the equations of fluid dynamics or continuum mechanics. Because of its close relation to the statistical ideas, Gingold and Monaghan [2] described the method as a Monte Carlo method, as did Lucy [3] who had, in effect, rediscovered the statistical technique. The basic SPH algorithm was improved to conserve linear and angular momentum exactly using the particle equivalent of the Lagrangian for a compressible nondissipative fluid [4].

In SPH, the function u(x) is represented using its information in a local domain via an integral form, i.e.,

$$u(\mathbf{x}) = \int_{-\infty}^{+\infty} u(\xi)\delta(\mathbf{x}-\xi)d\xi \qquad (1)$$

where $\mathbf{x} \in R^{\nu}$ (ν is the spatial dimension) and $\delta(\mathbf{x})$ is the Dirac delta function

The representation (1) is difficult to use for numerical analysis, though it is exact. So u(x) is approximated by the finite integral form of representation as follows [2, 3].

$$u^h(\mathbf{x}) = \int_{\Omega} u(\xi)W(\mathbf{x}-\xi,h)d\xi \qquad (2)$$

where W(x-ξ,h) is called weight function or kernel or smoothing function, his the smoothing length, which controls the size of the compact support domain Ω whose name is the influence domain. A finite integral representation (2) is valid and converges when the weight function satisfied these properties as follows [1, 5].

$W(\mathbf{x},h) > 0$ for $x \in \Omega$ (Positivity) (3a)

$W(\mathbf{x},h) = 0$ for $\forall x \in R^{\nu}/\Omega$ (Compact) (3b)

$\int_{\Omega} u(\xi)W(\mathbf{x}-\xi,h)d\xi=1$ (Unity) (3c)

$W(\mathbf{x},h) \to \delta(\mathbf{x})$ as $h \to 0$ (Dalta function behavior) (3d)

$W(\mathbf{x},h) \in C^p(\Omega), p \geq 1$ (Smooth) (3e)

It is noted that the property (3a) is not necessary mathematically but are important to ensure meaningful presentation of some physical phenomena.However, properties (3b) and (3c) are the minimum requirements. The fourth property ensures the method is converging to its exact form (1).

The last requirement comes from the essential point that we should construct a differentiable interpolation of a function from its values at the particles (interpolation points) by using a differentiable weight function. Then the derivatives of this interpolation can be obtained by ordinary differentiation and there is no need to use finite differences and no need for a grid. The continuous of the derivative of the weight function can prevent a large fluctuation in the force felt. This also gives rise to the name smoothed particle hydrodynamics. For details, we recommend [1, 5, 6] and the references therein.

It's important to chose weight functions in meshless methods. They should be constructed according to the demanded properties (3a)-(3e). After normalized for one dimension, we denote W(x-ξ,h) as W(r) where r=[x-ξ]/d_W,|.| denotes the Euclidean norm, positive numbers d_W are related to h and it can be different from point to point in general. It proves convenient to choose W(r) to be an even function [2]. Some commonly used weight functions include the Gaussian weight function[2], the cubic spline weight function [9], the quartic spline weight function [7] and so on. In practice, when choosing an appropriate weight function for one's SPH code, the main considerations are the order of interpolation, the number of nearest neighbours, the symmetry and stability properties. The readers interested in it are referred to find more references in [1]. In this paper, we propose a new weight function that satisfies the conditions (3a)-(3e). Furthermore, we investigate its consistency and analyze the estimation errors of the SPH codes with the weight function.

1. The cubic spline weight function（W_1） [10] :

$$W_1(d)=\begin{cases}\frac{2}{3}-4d^2+4d^3 & \text{for } d\le\frac{1}{2}\\ \frac{4}{3}-4d+4d^2-\frac{4}{3}d^3 & \text{for } \frac{1}{2}<d\le 1\\ 0 & \text{for } d>1\end{cases}$$

2. The quartic spline weight function (W_2)[10]:

$$W_2(d)=\begin{cases}1-6d^2+8d^3-3d^4 & \text{for } d\le 1\\ 0 & \text{for } d>1\end{cases}$$

3. The exponential weight function (W_3) [10]:

$$.W_3(d)=\begin{cases}e^{-(\frac{d}{\alpha})^2} & \text{for } d\le 1\\ 0 & \text{for } d>1\end{cases}$$

where α is a parameter.

II. A NEW INFINITY DIFFERENTIABLE WEIGHT FUNCTION

Inspired by the idea of the smeared-out Heaviside function in the level set method [8], we propose a new weight function which is infinity differentiable and satisfies the demanded properties (3a)-(3e).

For Heaviside function H(x) defined as

$$H(x)=\begin{cases}1; & \text{if } x>0,\\ 0\,; & \text{if } x\le 0\end{cases}$$

we can define a smeared-out Heaviside function

$$H_\varepsilon(x)=\frac{1}{\pi}\arctan\frac{x}{\varepsilon}+\frac{1}{2}$$

Where ε is a positive infinitely small parameter. The approximation concerning the different parameters ϵ are showed in Fig.1. We can see that the parameter ϵ influence the degree of smeared-out.

Denotes $\delta_\varepsilon(x)$ the derivative of $H_\varepsilon(x)$ on x, then we obtain

$$\delta_\varepsilon(x)=\frac{1}{\pi}\frac{\varepsilon}{1+x^2}$$

After normalized for one dimension, a new weight function $W_\varepsilon(r)$ is defined as

$$W_\varepsilon(r)=\begin{cases}\frac{1}{4\pi d_w^3}\delta_\varepsilon(r) & ,\text{if } 0\le r\le 1;\\ 0, & \text{else.}\end{cases}\qquad(4)$$

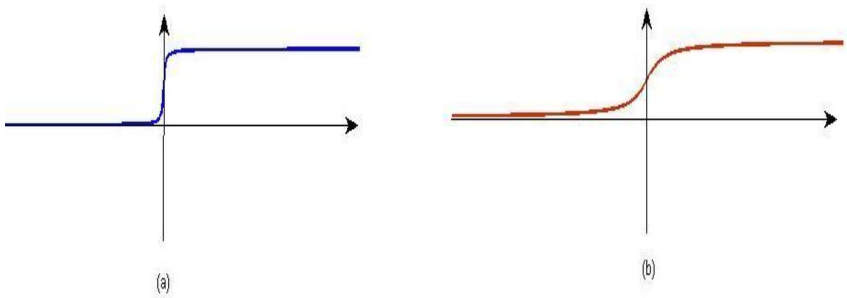

Fig.1 (a) ε = 0.01; (b) ε= 0.1

It is obvious that the weight function $W_\epsilon(r)$ has compact support, that is, the interactions are exactly zero for $|\mathbf{x} - \xi| > d_W$ This is a great computational advantage, since a potentially small number of neighbouring particles are the only contributors in the sums over the particles.

The function $W_\epsilon(r)$ is infinity differentiable. So $W_\epsilon(r)$ is not sensitive to the disorder of the particles. That is, the errors in approximating the integral interpolation by summation interpolation are small provided the particle disorder is not too large [6, 9].

The continuous consistency conditions result in a generalized approach to construct analytical smoothing functions that play a key role in the SPH formulation. The numerical solution obtained by the meshless method must converge to the true one when the nodal spacing approaches zero. So the shape functions have to satisfy a certain degree of consistency, which is achieved by properly choosing the weight function [1]. The term of consistency is used to measure the degree of approximation. In general, if the approximation can produce a polynomial of up to k-th order exactly, the approximation is said to have k-th order consistency, or C^k consistency.

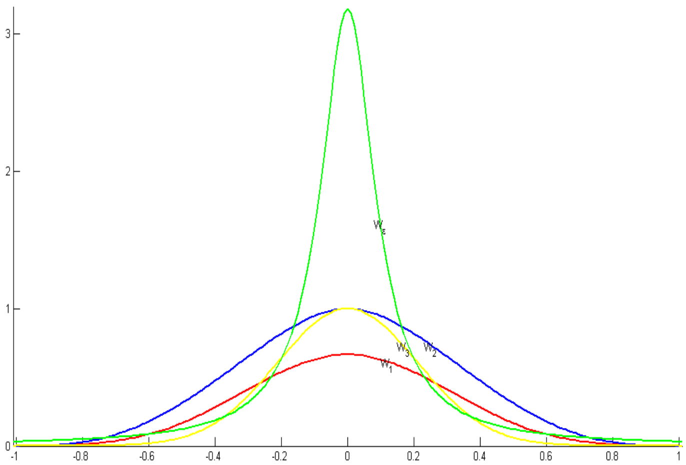

Fig. 2 weight functions: W_1:ubic spline weight function; W_2: quartic spline weight function; W_3: exponential weight function; W_ε: the new weight function.

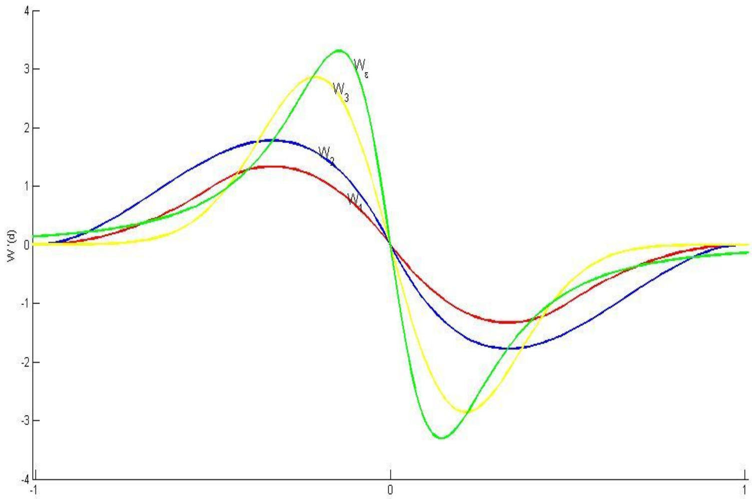

Fig. 3 The first derivatives of weight functions: W_1:cubic spline weight function; W_2:quartic spline weight function; W_3:exponential weight function; W_ε: the new weight function..

Before proceeding to the consistency of $W_\varepsilon(r)$, we first investigate the unity of $W_\varepsilon(r)$ Since there is a parameter ε involving in function $W_\varepsilon(r)$, so we introduce the definition of asymptotic unity.

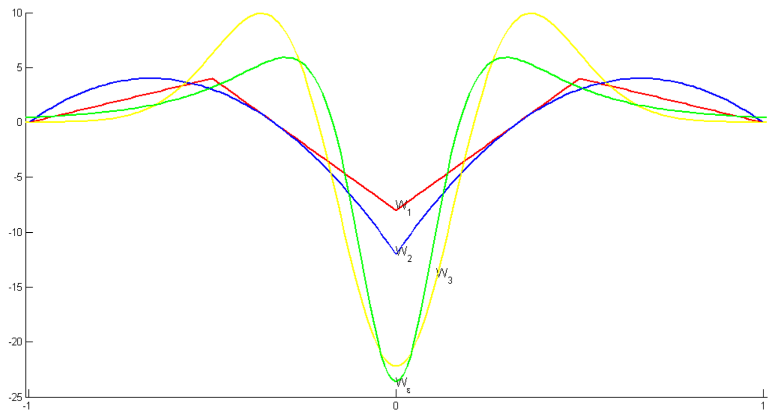

Fig. 4 The second derivatives of weight functions: W_1: cubic spline weight function; W_2: quartic spline weight function; W_3: exponential weight function; W_ε:: the new weight function**.**

Definition 1. We call a weight function W(x, h) is asymptotic unity if the equation as follows holds.

$$\lim_{h\to 0}\int_{\Omega} u(\xi)W(\mathbf{x}-\xi,h)d\xi = 1$$

By tedious but simple integral operation, we find that function$W_\varepsilon(r)$ is asymptotic unity when $\varepsilon = O(h)$. It is clear that the property of unity mean the lowest C^0 consistency. So we can see that the SPH approximation with weight function $W_\varepsilon(r)$ possess C^0 consistency in asymptotic sense.

In general, SPH does not possess C^1 consistency if the weight function satisfies only the properties (3a)-(3e). Furthermore, the condition that the weight function has to satisfy C^1 consistency is given in the following equation[1].

$$\mathbf{x} = \int_{\Omega} \xi W(\mathbf{x}-\xi,h)d\xi$$

As for the weight function$W_\varepsilon(r)$, we can now prove the following Theorem.

Theorem 1. when the parameter ε and d_W is proportional to h, the new weight function $W_\varepsilon(r)$ possesses C^1 consistency in the asymptotic sense, that is, the equation as follows holds.

$$\mathbf{x} = \lim_{h\to 0}\int_{\Omega} \xi W(\mathbf{x}-\xi,h)d\xi$$

Proof Denoting I_ε the first moment of the function and supposing $\mathbf{x}-\xi \in (a_x, b_x)$, where a_x and b_x are two variables which related to the point **x**, then

$$\begin{aligned} I_\varepsilon &= \frac{1}{\pi}\int_{a_x}^{b_x} \frac{\varepsilon(x-\xi)}{\varepsilon^2 + \frac{(x-\xi)^2}{d_W^2}} d\xi \\ &= -\frac{\varepsilon d_W^2}{2\pi}\int_{a_x}^{b_x} \frac{1}{\varepsilon^2 + \frac{(x-\xi)^2}{d_W^2}} d\frac{(x-\xi)^2}{d_W^2} \\ &= -\frac{\varepsilon d_W^2}{2\pi}\ln\left(\frac{\varepsilon^2 d_W^2 + b_x^2}{\varepsilon^2 d_W^2 + a_x^2}\right) \\ &= -c_1\frac{h^4}{2\pi}\ln\left(\frac{h^4 + c_2^2}{h^4 + c_3^2}\right) \to 0 \ \ (h \to 0) \end{aligned}$$

where c_1 is a constant related to $\varepsilon = O(h^2)$ and $d_W = O(h)$,$c_2, c_3 \in (-1,1)$and expresses the distance between points ξ and **x**.

Last, we analyzes the estimation errors of the SPH interpolation with the weight function $W_\varepsilon(r)$. In [6], the author stated that the errors are proportional to h^2 when the weight function is an even function of $\mathbf{x}-\xi$ since all odd moments are eliminated. Simply on account of its symmetry of $W_\varepsilon(r)$, so the function can interpolate to order h^2 accuracy. Moreover, a weight function is accurate to $O(h^4)$ when it satisfies the equation $\int \mathbf{x}^2 W(\mathbf{x}, h)d\mathbf{x}=0$

Following the evaluation of integral in asymptotic unity and C^1 consistency, we can find that the interpolation with function $W_\varepsilon(r)$ is accurate to $O(h^4)$.

III. CONCLUSIONS

An infinity differentiable weight function, which satisfies the demanded properties, is developed in this paper. The C^1 consistency and the $O(h^4)$ order estimation errors are also illustrated in the paper.

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