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Dynamics in a Discrete Prey-Predator System

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Abstract:- The stability analysis around equilibrium of a discrete-time predator prey system is considered in this paper. We obtain local stability conditions of the system near equilibrium points. The phase portraits are obtained for different sets of parameter values. Also limit cycles and bifurcation diagrams are provided for selective range of growth parameter. It is observed that prey and predator populations exhibit chaotic dynamics. Numerical simulations are performed and they exhibit rich dynamics of the discrete model. 2010 Mathematics Subject Classification. 39A30, 92D25

Keywords and phrases:- Difference equations, predator – prey system, fixed points, stability.

I. INTRODUCTION

Dynamics of interacting biological species has been studied in the past decades. The first models were put forward independently by Alfred Lotka (an American biophysicist, 1925) and Vito Volterra (an Italian Mathematician, 1926). Volterra formulated the model to explain oscillations in fish populations in Mediterranean. The model is based on the following assumptions:
(a) Prey population grow in an unlimited way when predator is absent
(b) Predators depend on the presence of prey to survive
(c) The rate of predation depends up on the likelihood that a predator encounters a prey
(d) The growth rate of the predator population is proportional to rate of predation.
The Lotka-Volterra model is the simplest model of predator-prey interactions, expressed by the following equations [2, 4].

$$x' = ax - bxy$$
$$y' = -cy + dxy$$

where *x, y*are the prey and predator population densities and *a, b, c,d* are positive constants.

II. MODEL DESCRIPTION AND EQUILIBRIUM POINTS

The discrete time models described by difference equations are more appropriate when the populations have non overlapping generations. Discrete models can also provide efficient computational models of continuous models for numerical simulations. The maps defined by simple difference equations can lead to rich complicated dynamics [1,3,5,7]. The paper [1] discusses the local stability of fixed points, bifurcation, chaotic behavior, Lyapunov exponents and fractal dimensions of the strange attractor associated with (1). This paper considers the following system of deference equations which describes interactions between two species and presents the various nature of fixed points and numerical simulations showing certain dynamical behavior.

$$\begin{aligned} x(n+1) &= rx(n)[1-x(n)] - ax(n)y(n) \\ y(n+1) &= -cy(n) + bx(n)y(n) \end{aligned} \qquad (1)$$

where *r,a,b,c>0* The system (1) has the equilibrium points $E_0=(0,0)$, $E_1 = \left(\frac{r-1}{r}, 0\right)$ and $E_2 = \left(\frac{1+c}{b}, \frac{r(b-1-c)}{ab} - \frac{1}{a}\right)$. The trivial equilibrium point E_0 corresponds to extinction of prey and predator species, E_1 corresponds to presence of prey and absence of predator and E_2 corresponds to coexistence of both species. The equilibrium point E_2 is an interior positive equilibrium point provided $r > \frac{b}{b-(1+c)}$.

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III. EIGEN VALUES AND STABILITY

Nonlinear systems are much harder to analyze than linear systems since they rarely possess analytical solutions. One of the most useful and important technique for analyzing nonlinear systems qualitatively is the analysis of the behavior of the solutions near equilibrium points using linearization. The local stability analysis of the model can be carried out by computing the Jacobian corresponding to each equilibrium point. The Jacobian matrix J for the system (1) is

$$J(x,y)=\begin{pmatrix} r-2rx-ay & -ax \\ by & bx-c \end{pmatrix}$$

The Jacobian at E_0 is of the form

$$J(E_0)=\begin{pmatrix} r & 0 \\ 0 & -c \end{pmatrix}$$

The eigen values are $\lambda_1=r$ and $\lambda_2=-c$. Stability is ensured if $|\lambda_{1,2}|<1$ which implies r < 1 and c < 1.The Jacobian matrix for E_1 is

$$J(E_1)=\begin{pmatrix} 2-r & a\left(\frac{1-r}{r}\right) \\ 0 & b\left(\frac{r-1}{r}\right)-c \end{pmatrix}$$

The eigen values are $\lambda_1=2-r$ and $\lambda_2=b\left(\frac{r-1}{r}\right)-c$ .The interior equilibrium point E_2 has the Jacobian

$$J(E_2)=\begin{pmatrix} 1-\frac{r(1+c)}{b} & -a\left(\frac{1+c}{b}\right) \\ \frac{r(b-1-c)-b}{a} & 1 \end{pmatrix}$$

Computation yields $Tr=2-\frac{r(1+c)}{b}$ and $Det=\frac{(1+c)r(b-2-c)}{b}-c$.

IV. CLASSIFICATION OF EQUILIBRIUM POINTS

The following lemma [8] is useful in the study of the nature of fixed points.

Lemma1. Let $p(\lambda)=\lambda^2-B\lambda+C$ andλ_1, λ_2 be the roots of $p(\lambda)=0$. Suppose that $p(1)>0$ Then we have

(i) $|\lambda_1|<1$ and $|\lambda_2|<1$ if and only if $p(-1)>0$ and $C<1$.

(ii) $|\lambda_1|<1$ and $|\lambda_2|>1$ (or $|\lambda_1|>1$ and $|\lambda_2|<1$) if and only if $p(-1)<0$.

(iii) $|\lambda_1|>1$ and $|\lambda_2|>1$ if and only if $p(-1)>0$ and $C>1$.

(iv) $|\lambda_1|=-1$ and $|\lambda_2|\neq 1$ if and only if $p(-1)=0$ and $B\neq 0,2$.

(v) λ_1 and λ_2 are complex and $|\lambda_1|=|\lambda_2|$ if and only if $B^2-4C<0$ and $C=1$.

The characteristic roots λ_1 and λ_2 of $p(\lambda)=0$ are called eigen values of the fixed point (x^*,y^*). The fixed point (x^*,y^*) is a sink if $|\lambda_{1,2}|<1$. Hence the sink is locally asymptotically stable. The fixed point (x^*,y^*) is a source if $|\lambda_{1,2}|>1$. The source is locally unstable. The fixed point (x^*,y^*) is a saddle if $|\lambda_1|>1$ and $|\lambda_2|<1$ (or $|\lambda_1|<1$ and $|\lambda_2|>1$). Finally (x^*,y^*) is called non hyperbolic if either $|\lambda_1|=1$ or $|\lambda_2|=1$. For the system (1), we have the following results.

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Proposition 2. The fixed point E_0 is a

- Sink if $r<1$ and $c<1$.
- Source if $r>1$ and $c>1$.
- Saddle if $r>1$ and $c<1$.
- Non hyperbolic if $r=1$ and $c=1$.

Proposition 3. The fixed point E_1 is a

- Sink if $1<r<3$ and $b<\dfrac{r(1+c)}{r-1}$.
- Source if $r>3$ and $b>\dfrac{r(1+c)}{r-1}$.
- Saddle if $1<r<3$ and $b>\dfrac{r(1+c)}{r-1}$.

Proposition 4. The fixed point E_2 is a

- Sink if $\dfrac{b(c-3)}{(1+c)(b-(3+c))}<r<\dfrac{b}{b-(2+c)}$.
- Source if $r>\dfrac{b(c-3)}{(1+c)(b-(3+c))}$ and $r>\dfrac{b}{b-(2+c)}$.
- Saddle if $r<\dfrac{b(c-3)}{(1+c)(b-(3+c))}$.

V. NUMERICAL SIMULATIONS

In this section, we provide the numerical simulations to illustrate some results of the previous sections. Mainly, we present the time plots of the solutions *x* and *y* with phase plane diagrams (around the positive equilibrium point) for the predator-prey systems. Dynamic natures of the system (1) about the equilibrium points with different sets of parameter values are presented. Existence of limit cycles for selective set of parameters is established through phase planes in *Figures-3, 4*. Also the bifurcation diagram, *Figure-5*, indicates the existence of chaos in both prey and predator populations.

Example1.*For the values r = 2.89, a = 0.099, b = 3.075, c = 1.09, we obtain E_1=(0.65, 0) which is an axial fixed point. Eigen values are λ_1 = -0.89 and λ_2 = 0.9209 so that $|\lambda_{1,2}|<1$. Hence the fixed point is stable. The time plot and the phase diagramillustrate the result, see Figure - 1***.**

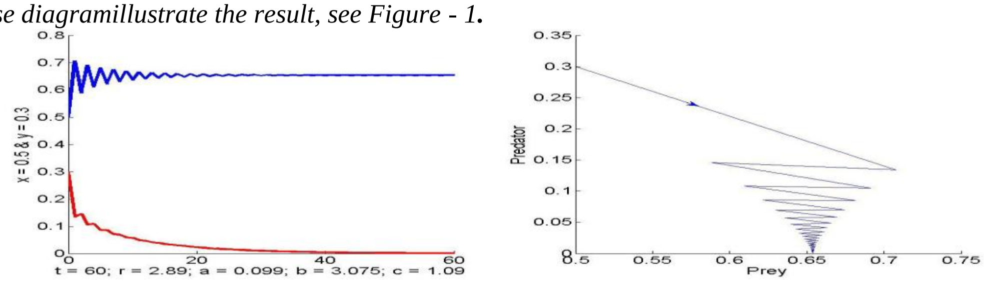


Figure 1. Stability at E_1

Example 2. *In this example, we take r = 2.41, a = 1.19, b = 3.91 and c = 0.45. Computations yield E_2 = (0.37, 0.43). The eigen values are $\lambda_{1,2}$ = 0.5531 ± i0.7409 and $\lambda_{1,2}$ = 0.9246 < 1. Hence the criteria for stability are satisfied [6]. The phase portrait in Figure - 2 shows a sink and the trajectory spirals towards the fixed point E_2.*

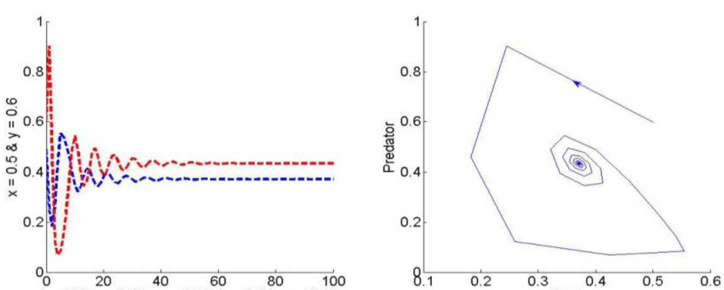


Figure 1. Stability at E_2

Example 3. *The parameters are r = 2.41, a = 1.43, b = 3.91, c = 0.25. The initial conditions on the populations of the species are x(0) = 0.2 and y(0) = 0.3. The trajectory spirals inwards but does not approach a point. The trajectory finally settles down as a limit cycle, see Figure-3.*

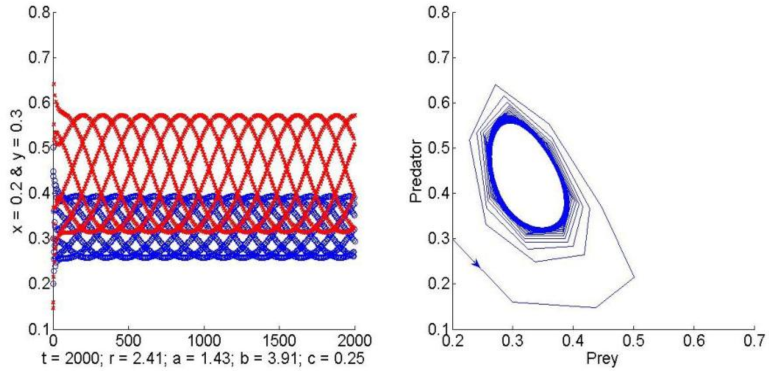
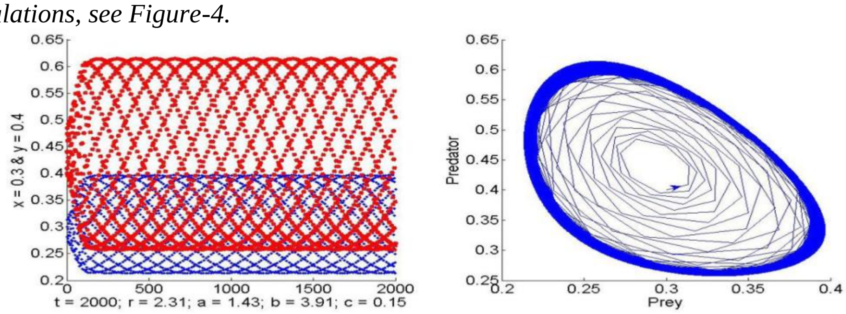


Figure 3. Limit Cycle-1

The model (1) with parameters r = 2.31, a = 1.43, b = 3.91, c = 0.15 and initial conditions x(0) = 0.3, y(0) = 0.4 exhibits another form of limit cycle. In this case the trajectory moves out in growing spirals and finally approaches the limit cycle. The existence of limit cycles for selective range of parameters shows the oscillating nature of the populations, see Figure-4.



VI. **Figure 4.** Limit Cycle – 2

Studies in population dynamics focuses on identifying qualitative changes in the long-term dynamics predicted by the model. Bifurcation theory deals with classifying, ordering and studying the regularity in the dynamical changes. Bifurcation diagrams provide information about abrupt changes in the dynamics and the values of parameters at which such changes occur. Also they provide information about the dependence of the dynamics on a certain parameter. Qualitative changes are tied with bifurcation.

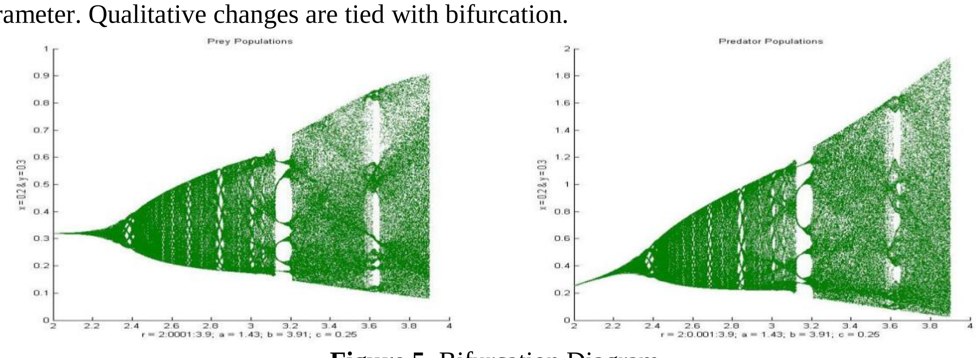


Figure 5. Bifurcation Diagram

Example 4. *The parameters are assigned the values a = 1.43, b = 3.91, c = 0.25 and the bifurcation diagram is plotted for the growth parameter in the range 2 - 3.9. Both prey and predator population undergoes chaos,Figure-5.*

This paper, dealt with a 2-dimensional discrete predator - prey system. Fixed points are found and stability conditions are obtained. The results are illustrated with suitable hypothetical sets of parameter values. Numerical simulations are presented to show the dynamical behavior of the system (1). Finally, bifurcation diagrams for both species are presented.

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