Fixed Point Theorems for Mappings under General Contractive Condition of Integral Type

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Abstract—In the present paper, we establish a fixed point theorem for a mapping and a common fixed point theorem for a pair of mappings for rational expression. The mapping involved here generalizes various type of contractive mappings in integral setting.

Keywords—Fixed point, Common Fixed point, complete metric space, Continuous Mapping, Compatible Mapping

I.   INTRODUCTION AND PRELIMINARIES

Impact of fixed point theory in different branches of mathematics and its applications is immense.The first important result on fixed point for contractive type mapping was the much celebrated Banach's contraction principle by S.Banach [1] in 1922. In the general setting of complete metric space,this theorem runs as follows(see Theorem 2.1,[4] or,Theorem 1.2.2,[10]) In the present paper we will find some fixed point and common fixed point theorems for rational expression, which will satisfy the well known results. I have developed new result in above theorem with rational expression, by using R.Bhardwaj,Some common fixed point theorem in Metric space using integral type mappings[12].

Theorem 1.1 (Banach's contraction principal)
Let (X,d) be a complete matric space,$c \in (0,1)$ and $f:X \to X$ be a mapping such that for each $x,y \in X$,
$$d(fx,fy) \le c\, d(x,y) \qquad\qquad \dots\dots\dots\dots\dots(1.1)$$

then f has a unique fixed point $a \in X$,such that for each $x \in X$, $\lim_{n \to \infty} f^n x = a$.

after this classical result , Kannaan [5] gave a substantially new contractive mapping to prove fixed point theorem.Since then a number of mathematicians have been working on fixed point theory dealing with mappings satisfying various type of contractive conditions(see[3],[5],[7],[8],[9] and [11] for details).
In 2002,A.Branciari [2] analyzed the existence of fixed point for mapping f defined on a complete matric space (X,d) satisfying a general contractive condition of integral type.

Theorem 1.2 (Branciari)
Let (X,d) be a complete matric space,$c \in (0,1)$ and $f:X \to X$ be a mapping such that for each $x,y \in X$,
$$\int_0^{d(fx,fy)} \varphi(t)dt \le c \int_0^{d(x,y)} \varphi(t)dt \qquad\qquad \dots\dots\dots\dots\dots(1.2)$$

Where $\varphi:[0,+\infty) \to [0,+\infty)$ is a Lesbesgue-integrable mapping which is summable(i.e.with finite integral) on each compact subset of $[0,+\infty)$,nonnegative,and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt$,then f has a unique fixed point $a \in X$,such that for each

$x \in X$, $\lim_{n \to \infty} f^n x = a$.

After the paper of Branciari, a lot of research works have been carried out on generalizing contractive conditions of integral type for different contractive mappings satisfying various known prpperties. A fine work has been done by Rhoades [6] extending the result of Branciari by replacing the condition (1.2) by the following

$$\int_0^{d(fx,fy)} \varphi(t)dt \le c \int_0^{\max\{d(x,y),d(x,fx),d(y,fy),\frac{[d(x,fy)+d(y,fx)]}{2}\}} \varphi(t)dt \qquad\qquad \dots\dots\dots\dots(1.3)$$

The aim of this paper is to generalize some mixed type of contractive conditions to the mapping and then a pair of mappings satisfying a general contractive condition of integral type,which includes several known contractive mappings such as Kannan type [5],Chatterjea type [3],Zamfirescu type [11],etc.

II.   MAIN RESULTS

Let f beaself mapping of a complete matric space (x,d) satisfying the following condition:

$$\int_0^{d(fx,fy)}\varphi(t)dt \le \alpha \int_0^{[d(x,fx)+d(y,fy)]}\varphi(t)dt + \beta \int_0^{d(x,y)}\varphi(t)dt + \gamma \int_0^{\max\{d(x,fy),d(y,fx)\}}\varphi(t)dt +$$

$$\delta \int_0^{\frac{d(x,fy)+d(y,fx)+d(x,fx)}{1+d(x,fy).d(y,fx).d(x,fx)}}\varphi(t)dt \qquad \ldots\ldots\ldots\ldots\ldots(2.1)$$

For each x,yϵX with non-negative reals $\alpha,\beta,\gamma,\delta,$ such that $0<2\alpha+\beta+2\gamma+3\delta<1,$

where $\varphi:R^+\to R^+$ is a Lesbesgue-integrable mapping which is summable(i.e.with finite integral) on each compact

subset of R^+ ,nonnegative,and such that for each $\epsilon>0, \int_0^\epsilon \varphi(t)dt$ ....(2.2)

Then f has a unique fixed point zϵX,such that for each xϵX, $\lim_{n\to\infty} f^n x = z.$

Proof: Let $x_0\epsilon$X and, for brevity,definex_n=fx_{n-1}.For each integer n≥1,from (2.1) we get,

$$\int_0^{d(Xn,Xn+1)}\varphi(t)dt \le \int_0^{d(fXn-1,fXn)}\varphi(t)dt$$

$$\le \alpha \int_0^{[d(Xn-1,Xn)+d(Xn,Xn+1)]}\varphi(t)dt + \beta \int_0^{d(Xn-1,Xn)}\varphi(t)dt + \gamma \int_0^{\max\{d(Xn-1,Xn+1),d(Xn,Xn)\}}\varphi(t)dt +$$

$$\delta \int_0^{\frac{d(Xn-1,Xn+1)+d(Xn,Xn)+d(Xn-1,Xn)}{1+d(Xn-1,Xn+1).d(Xn,Xn).d(Xn-1,Xn)}}\varphi(t)dt$$

$$\le \alpha \int_0^{[d(Xn-1,Xn)+d(Xn,Xn+1)]}\varphi(t)dt + \beta \int_0^{d(Xn-1,Xn)}\varphi(t)dt + \gamma \int_0^{d(Xn-1,Xn+1)}\varphi(t)dt +$$

$$\delta \int_0^{d(Xn-1,Xn+1)+d(Xn-1,Xn)}\varphi(t)dt$$

$$\le (\alpha+\beta+\gamma+2\delta)\int_0^{d(Xn-1,Xn)}\varphi(t)dt + (\alpha+\gamma+\delta)\int_0^{d(Xn,Xn+1)}\varphi(t)dt$$

Which implies that

$$\int_0^{d(Xn,Xn+1)}\varphi(t)dt \le (\frac{\alpha+\beta+\gamma+2\delta}{1-\alpha-\gamma-\delta})\int_0^{d(Xn-1,Xn)}\varphi(t)dt$$

And so, $\int_0^{d(Xn,Xn+1)}\varphi(t)dt \le h \int_0^{d(Xn-1,Xn)}\varphi(t)dt$ where   h=$\frac{\alpha+\beta+\gamma+2\delta}{1-\alpha-\gamma-\delta}$ <1   …………..(2.3)

Thus by routine calculation,

$$\int_0^{d(Xn,Xn+1)}\varphi(t)dt \le h^n \int_0^{d(X_0,X_1)}\varphi(t)dt$$

…………….(2.4)

Taking limit of (2.4) as n→∞, we get $\lim_n \int_o^{d(Xn,Xn+1)}\varphi(t)dt = 0$

Which,from (2.2) implies that $\lim_n d(Xn,Xn+1)=0$ …………..(2.5)

We now show that {Xn} is a Cauchy sequence. Suppose that it is not. Then there exists an $\epsilon > 0$ and subsequences {m(p)} and {n(p)} such that m(p)<n(p)<m(p+1) with

$$d(Xm(p),Xn(P)) \geq \epsilon, \quad d(Xm(p),Xn(P)-1) < \epsilon \qquad \qquad \qquad \text{..............(2.6)}$$

$$d(Xm(p)-1,Xn(P)-1) \leq d(Xm(p)-1,Xm(P)) + d(Xm(p),Xn(P)-1)$$
$$< d(Xm(p)-1,Xm(P)) + \epsilon \qquad \qquad \text{.............(2.7)}$$

Hence $\quad \lim\limits_{p} \int\limits_{o}^{d(Xm(p)-1,,Xn(p)-1)} \varphi(t)dt \leq \int\limits_{o}^{\in} \varphi(t)dt$

$$\text{..............(2.8)}$$

Using (2.3),(2.6) and (2.8) we get

$$\int\limits_{o}^{\in} \varphi(t)dt \leq \int\limits_{o}^{d(Xm(p),,Xn(p))} \varphi(t)dt \leq h \int\limits_{o}^{d(Xm(p)-1,,Xn(p)-1)} \varphi(t)dt \leq h \int\limits_{o}^{\in} \varphi(t)dt$$

Which is a contradiction, since $h\epsilon(0,1)$. Therefore {Xn} is Cauchy, hence convergent. Call the limit z. From (2.1) we get

$$\int\limits_{o}^{d(fz,Xn+1)} \varphi(t)dt \leq \alpha \int\limits_{0}^{[d(z,fz)+d(Xn,Xn+1)]} \varphi(t)dt + \beta \int\limits_{0}^{d(z,Xn)} \varphi(t)dt + \gamma \int\limits_{0}^{\max\{d(z,Xn+1),d(Xn,fz)\}} \varphi(t)dt \; +$$

$$\delta \int\limits_{0}^{\frac{d(z,Xn+1)+d(Xn,fz)+d(z,fz)}{1+d(z,Xn+1).d(Xn,fz).d(z,fz)}} \varphi(t)dt$$

Taking limit as n→∞, we get

$$\int\limits_{o}^{d(fz,z)} \varphi(t)dt \leq (\alpha + \gamma + 2\delta) \int\limits_{0}^{d(z,fz)} \varphi(t)dt$$

As

$$2\alpha + \beta + 2\gamma + 3\delta <1, \qquad \int\limits_{o}^{d(fz,z)} \varphi(t)dt = 0$$

Which, from (2.2), implies that $d(fz,z)=0$ or $fz=z$.

Next suppose that w $(\neq z)$ be another fixed point of f. Then from (2.1) we have

$$\int\limits_{o}^{d(z,w)} \varphi(t)dt = \int\limits_{o}^{d(fz,fw)} \varphi(t)dt$$

$$\int\limits_{o}^{d(z,w)} \varphi(t)dt \leq \alpha \int\limits_{0}^{[d(z,fz)+d(w,fw)]} \varphi(t)dt + \beta \int\limits_{0}^{d(z,w)} \varphi(t)dt + \gamma \int\limits_{0}^{\max\{d(z,fw),d(w,fz)\}} \varphi(t)dt +$$

$$\delta \int\limits_{0}^{\frac{d(z,fw)+d(w,fz)+d(z,fz)}{1+d(z,fw).d(w,fz).d(z,fz)}} \varphi(t)dt$$

$$\leq (\beta + \gamma + 2\delta) \int\limits_{0}^{d(z,w)} \varphi(t)dt$$

Since,

$$\beta + \gamma + 2\delta <1 \text{ this implies that, ,} \qquad \int\limits_{o}^{d(z,w)} \varphi(t)dt = 0$$

Which from (2.2), implies that d(z,w)=0 or, z=w and so the fixed point is unique.

Remark. On setting $\varphi(t) = 1$ over R$^+$, the contractive condition of integral type transforms into a general contractive condition not involving integrals.

Remark. From condition (2.1) of integraltype,several contractive mappings of integral type can be obtained.

1. $\beta = \gamma = \delta = 0$ and $\alpha \in (0, \frac{1}{2})$ gives Kannanof integral type.

2. $\alpha = \beta = \delta = 0$ and $\gamma \in (0, \frac{1}{2})$ gives Chatterjea [3] map of integral type.

Theorem 2.3. Let f and g be self mappings of a complete metric space (X,d) satisfying the following condition:

$$\int_o^{d(fx,gy)} \varphi(t)dt \leq \alpha \int_0^{[d(x,fx)+d(y,gy)]} \varphi(t)dt + \beta \int_0^{d(x,y)} \varphi(t)dt + \gamma \int_0^{\max\{d(x,gy),d(y,fx)\}} \varphi(t)dt +$$

$$\delta \int_0^{\frac{d(x,gy)+d(y,fx)+d(x,fx)}{1+d(x,gy).d(y,fx).d(x,fx)}} \varphi(t)dt$$

............(2.9)

For each x,y$\in$X with nonnegative reals $\alpha, \beta, \gamma, \delta$ such that $2\alpha + \beta + 2\gamma + 3\delta < 1$, where φ: $R^+ \to R^+$ is a Lesbesgue-integrable mapping which is summable(i.e.with finite integral) on each compact subset of R^+ ,nonnegative,and such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)dt$ ....(2.10)

Then f and g have a unique common fixed point z$\in$X.

Proof. Let $x_0 \in X$ and, for brevity,define $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$. For each integer n$\geq$0, from (2.9) we get,

$$\int_0^{d(X2n+1,X2n+2)} \varphi(t)dt \leq \int_0^{d(fX2n,gX2n+1)} \varphi(t)dt$$

$$\leq \alpha \int_0^{[d(X2n,X2n+1)+d(X2n+1,X2n+2)]} \varphi(t)dt + \beta \int_0^{d(X2n,X2n+1)} \varphi(t)dt + \gamma \int_0^{\max\{d(X2n,X2n+2),d(X2n+1,X2n+1)\}} \varphi(t)dt +$$

$$\delta \int_0^{\frac{d(X2n,X2n+2)+d(X2n+1,X2n+1)+d(X2n,X2n+1)}{1+d(X2n,X2n+2).d(X2n+1,X2n+1).d(X2n,X2n+1)}} \varphi(t)dt$$

$$\leq (\alpha + \beta + \gamma + 2\delta) \int_0^{d(X2n,X2n+1)} \varphi(t)dt + (\alpha + \gamma + \delta) \int_0^{d(X2n+1,X2n+2)} \varphi(t)dt$$

Which implies that

$$\int_0^{d(X2n+1,X2n+2)} \varphi(t)dt \leq \left(\frac{\alpha+\beta+\gamma+2\delta}{1-\alpha-\gamma-\delta}\right) \int_0^{d(X2n,X2n+1)} \varphi(t)dt$$

And so, $\int_0^{d(X2n+1,X2n+2)} \varphi(t)dt \leq h \int_0^{d(X2n,X2n+1)} \varphi(t)dt$ where h=$\frac{\alpha+\beta+\gamma+2\delta}{1-\alpha-\gamma-\delta} < 1$ ....(2.11)

Similarly $\int_0^{d(X2n,X2n+1)} \varphi(t)dt \leq h \int_0^{d(X2n-1,X2n)} \varphi(t)dt$

.............(2.12)

Thus in general, for all n=1,2,..............

$$\int_0^{d(Xn,Xn+1)} \varphi(t)dt \leq h \int_0^{d(Xn-1,Xn)} \varphi(t)dt$$

.............(2.13)

Thus by routine calculation, we have

$$\int_0^{d(Xn,Xn+1)} \varphi(t)dt \le h^n \int_0^{d(X_0,X_1)} \varphi(t)dt$$

Taking limit as n→∞, we get

$$\lim_n \int_o^{d(Xn,Xn+1)} \varphi(t)dt = 0$$

Which, from (2.10) implies that

$$\lim_n d(Xn, Xn+1) = 0 \qquad\qquad ....(2.14)$$

We now show that {Xn} is a Cauchy sequence. Suppose that it is not. Then there exists an ε>0 and subsequences {2m(p)} and {2n(p)} such that p<2m(p)<2n(p) with

$$d(X2m(p),X2n(P))\ge\epsilon,\ d(X2m(p),X2n(P)\text{-}2)<\epsilon \qquad ....(2.15)$$

Now

$$d(X2m(p),X2n(P))\le d(X2m(p),X2n(P)\text{-}2)+ d(X2n(p)\text{-}2,X2n(P)\text{-}1)+ d(X2n(p)\text{-}1,X2n(P))$$
$$<\epsilon+d(X2n(p)\text{-}2,X2n(P)\text{-}1)+ d(X2n(p)\text{-}1,X2n(P)) \qquad ....(2.16)$$
$$..............(2.7)$$

Hence

$$\lim_p \int_o^{d(X2m(p),X2n(p))} \varphi(t)dt \le \int_o^{\epsilon} \varphi(t)dt$$

$$..............(2.17)$$

Then by (2.13) we get

$$\int_o^{d(X2m(p),,X2n(p))} \varphi(t)dt \le h \int_o^{d(X2m(p)-1,,X2n(p)-1)} \varphi(t)dt$$

$$\le h[\int_o^{d(X2m(p)-1,,X2m(p))} \varphi(t)dt\ +\ \int_o^{d(X2m(p),,X2n(p))} \varphi(t)dt+\ \int_o^{d(X2n(p)-1,X2n(p))} \varphi(t)dt]$$

Taking limit as p→∞ we get

$$\int_0^{\epsilon} \varphi(t)dt \le h\int_0^{\epsilon} \varphi(t)dt$$

Which is a contradiction, since hε(0,1). Therefore {Xn} is Cauchy, hence convergent. Call the limit z.
From (2.9) we get

$$\int_o^{d(fz,X2n+2)} \varphi(t)dt = \int_0^{d(fz,gX2n+1)} \varphi(t)dt$$

$$\le \alpha \int_0^{[d(z,fz)+d(X2n+1,X2n+2)]} \varphi(t)dt\ +\ \beta \int_0^{d(z,X2n+1)} \varphi(t)dt + \gamma \int_0^{\max\{d(z,X2n+2),d(X2n+1,fz)\}} \varphi(t)dt\ +$$

$$\delta \int_0^{\frac{d(z,X2n+2)+d(X2n+1,fz)+d(z,fz)}{1+d(z,X2n+2).d(X2n+1,fz).d(z,fz)}} \varphi(t)dt$$

Taking limit as n→∞, we get

$$\int_o^{d(fz,z)} \varphi(t)dt \le (\alpha + \gamma + 2\delta) \int_0^{d(z,fz)} \varphi(t)dt$$

As

$$2\alpha + \beta + 2\gamma + 3\delta <1, \qquad \int_o^{d(fz,z)} \varphi(t)dt = 0$$

Which, from (2.10), implies that d(fz,z)=0 or fz=z. Similarly it can be shown that gz=z. So f and g have a common fixed

point zϵX. We now show that z is the unique common fixed point of f and g.Ifnot,then let

w ($\neq$ z) be another fixed point of f and g. Then from (2.9) we have

$$\int\limits_{o}^{d(z,w)}\varphi(t)dt = \int\limits_{o}^{d(fz,gw)}\varphi(t)dt$$

$$\int\limits_{o}^{d(z,w)}\varphi(t)dt \leq \alpha \int\limits_{0}^{[d(z,fz)+d(w,gw)]}\varphi(t)dt + \beta \int\limits_{0}^{d(z,w)}\varphi(t)dt + \gamma \int\limits_{0}^{\max\{d(z,gw),d(w,fz)\}}\varphi(t)dt +$$

$$\delta \int\limits_{0}^{\frac{d(z,gw)+d(w,fz)+d(z,fz)}{1+d(z,gw).d(w,fz).d(z,fz)}}\varphi(t)dt$$

$$\leq (\beta+\gamma+2\delta) \int\limits_{0}^{d(z,w)}\varphi(t)dt$$

Since,

$$\beta+\gamma+2\delta <1 \quad \text{this implies that, ,} \quad \int\limits_{o}^{d(z,w)}\varphi(t)dt = 0$$

Which from (2.10), implies that d(z,w)=0 or, z=w and so the fixed point is unique.

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