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Efficient dominating sets in ladder graphs

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***Abstract*—An independent set S of vertices in a graph is an efficient dominating set when each vertex not in S is adjacent to exactly one vertex in S. In this note, we prove that a ladder graph L_n has an efficient dominating set if and only if n be a multiple of 4. Also we determine the domination number of a ladder graph L_n.**

***Keywords*—Efficient dominating set, Perfect code, Domination number,Ladder graph,Sphere packing**

I. INTRODUCTION

Let$\Gamma = (V, E)$ be a finite undirected graph with no loops and multiple edges. We follow the terminology of [2] and [3]. Given $S \subseteq V$, let the open neighbourhood $N(S)$ of Sin Γ be the subset of vertices in $V\backslash S$ adjacent to some vertex in S, and let the corresponding closed neighbourhood be $N[S] = N(S) \cup S$. A set $S \subset V$ is a dominating set if $N[S] = V$, that is, every vertex in $V\backslash S$ is adjacent to some vertex in S. The domination number $\gamma(\Gamma)$is the minimum cardinality of a dominating set inΓ. If the dominating set S is a stable set of Γ, then S is an independent dominating set. Also, when every vertex in $V\backslash S$ is adjacent to exactly one vertex in S, then S is a perfect dominating set. A dominating set S which is both independent and perfect is an efficient dominating set. In what follows we may refer to an efficient dominating set as an E-set. Given the graph $\Gamma = (V, E)$, we call any subset C of V a code in Γ.

We say that C corrects t errors if and only if the sets S_c=$\{u | u \epsilon V, d(u, c) \leq t\}$ are pairwise disjoint. Morever we call C a t-perfect code if and only if these sets form a partition of V. A t-perfect code C is called nontrivial if and only if $t > 0$ and $card(C) > 1$.

E-sets correspond to perfect 1-correcting codes in Γ,as treated by Biggs[1] and Kratochvil[4]. Equivalently, they provide a perfect packing of Γ by balls of radius 1. When Γ is r-regular, the so-called sphere packing condition

$$|V| = (r + 1)|C|$$

is trivially a necessary condition for C to be an E-set of Γ.

We need the following two results from [3]. We recall that the maximum degrees of vertices of Γ is denoted by $\Delta(\Gamma)$.

Theorem 1.1*For any graph Γ of order n,*

$$\left\lceil \frac{n}{1 + \Delta(\Gamma)} \right\rceil \leq \gamma(\Gamma) \leq n - \Delta(\Gamma)$$

Theorem 1.2 *If Γhas an efficient dominating set, then the cardinality of any efficient dominating set equals the domination number $\gamma(\Gamma)$. In particular, all efficient dominating sets of Γhave the same cardinality.*

II. THE MAIN RESULTS

We use the following remark in the sequel

Remark. Suppose that $n(> 1)$be a natural number, the dihedral group D_{2n} of order $2n$ is defined by thepresentation

$$D_{2n} = < x, y | x^n = y^2 = (xy)^2 = 1 >.$$

Then the distinct elements of D_{2n} are as follows:

$$\{1, x, \dots, x^{n-1}, y, xy, \dots, x^{n-1}y\}.$$

The Cayley graph $C(D_{2n}, X)$ of D_{2n} with respect to the generating set$X = \{x, x^{-1}$,y} is a ladder graph L_n.

Theorem 2.1 *Let L_n be a ladder graph of order $2n$. Then the domination number of L_n is determined as follows:*

$$\gamma(L_n) = \begin{cases} \frac{n}{2} & 4|n \\ \frac{n}{2} + 1 & if 2|n, 4 \nmid n \\ \left\lceil \frac{n}{2} \right\rceil + 1 & 2 \nmid n \end{cases}$$

Proof. If $4|n$, then $n = 4k$ for some natural number k. Let $S = \{1, x^4, x^8, \dots, x^{4k-4}, x^2y, x^6y, \dots, x^{4k-2}y\}$. We show that S is a dominating set. It is easy to see that $|S| = \frac{n}{2} = 2k$. Since $L_n \cong C(D_{2n}, X)$ then for all i, $2 \le i \le n-1$, a vertex x^i is adjacent to three vertices x^{i+1}, x^{i-1} and x^iy. There are 3 cases.

If $4|i$, then $x^i \in S$.

If $2|i$ and $4 \nmid i$, then $x^iy \in S$.

If $2 \nmid i$, then $i = 4m-1$ or $i = 4m+1$ for some natural number m.

If $i = 4m-1$, then $x^{i+1} \in S$ and if $i = 4m+1$, then $x^{i-1} \in S$.

Similarly, for all $i, 1 \le i \le n-1$, a vertex x^iy is adjacent to some vertex in S. For three vertices $1, x^{n-1}, y$, we have $1 \in S$ and y, x^{n-1} are adjacent to 1. So S is a dominating set. Since L_n is a 3-regular graph, it follows from sphere packing condition that

$$\gamma(L_n) \ge \frac{n}{2}$$

Also S is a dominating set of L_n which $|S| = \frac{n}{2}$. Hence it follows that $\gamma(L_n) = \frac{n}{2}$; $(n = 4k)$.

If n be odd, then we have 2 cases.

Case1: $n = 4k+1$ for some natural number k.

In this case, let $S = \{1, x^4, x^8, \dots, x^{4k}, x^2y, x^6y, \dots x^{4k-2}y\}$, then

$$|S| = 2k+1 = \left\lceil \frac{4k+1}{2} \right\rceil + 1.$$

It is easy to see that S is a dominating set, and by theorem 1.1, $\left\lceil \frac{n}{2} \right\rceil + 1 \le \gamma(L_n) \le |S| = \left\lceil \frac{n}{2} \right\rceil + 1$.

Therefore, $\gamma(L_n) = \left\lceil \frac{n}{2} \right\rceil + 1$;(n=4k+1).

Case2: $n = 4k-1$ for some natural number k.

Let $S = \{1, x^4, x^8, \dots, x^{4k-4}, x^2y, x^6y, \dots x^{4k-2}y\}$, then

$$|S| = 2k = [\frac{4k-1}{2}]+1=[\frac{n}{2}] + 1.$$

It is similar to case1. So $\gamma(L_n) = \left\lceil \frac{n}{2} \right\rceil + 1$; $(n = 4k-1)$.

If $2|n$ and $4 \nmid n$, then $n = 2k$ such that k is an odd number. Let

$$S = \{1, x^4, x^8, \dots, x^{2k-2}, x^2y, x^6y, \dots, x^{2k-4}y, x^{2k-1}\}.$$

It is easy to see that S is a dominating set and $|S| = k+1 = \frac{n}{2} + 1$.

By theorem 1.1,

$$\frac{n}{2} \le \gamma(L_n) \le |S| = \frac{n}{2} + 1.$$

We show that the domination number $\gamma(L_n) \ne \frac{n}{2}$. It follows from the proof of theorem 1.1, that $\gamma(\Gamma) = \frac{n}{1+\Delta(\Gamma)}$ if and only if Γ has a γ-set(dominating set of minimum cardinality) S such that $N[u] \cap N[v] = \emptyset$ for all $u, v \in S$ and $|N[v]| = \Delta(\Gamma)$ for all $v \in S$. If $\gamma(L_n) = \frac{n}{2}$, then L_n has a γ-set S such that $N[u] \cap N[v] = \emptyset$ for all $u, v \in S$. Since $N[S] = V(L_n)$, therefore

$$3k = |N[S]| = |V(L_n)| = 2\text{n} = 4\text{k}$$

is a contradition. So $\gamma(L_n) \ne \frac{n}{2}$. ■

Remark. If the ladder graph L_n, which is a 3-regular graph has an efficient dominating set S, then by the sphere packing condition we have $|S| = \frac{n}{2}$. Therefore, only the ladder graph with even order has an efficient dominating set.

Theorem 2.2 *The ladder graph L_n of order $2n$, has an efficient dominating set if and only if $4|n$.*

Proof. By remark and this fact that $|S| = \gamma(L_n)$, the ladder graph has an E-set if $4|n$. Also if $4|n$, then we show that the set

$$S = \{1, x^4, x^8, \dots, x^{4k-4}, x^2y, x^6y, \dots, x^{4k-2}y\}$$

is a dominating set. It is easy to see that S is an independent and perfect set. Therefore S is an E-set of ladder graph L_n. ■

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