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Torsional Vibrations of Thin-Walled Open Section Beams with Unequal Rotational Restraints at the Ends using Spectral Dynamics Approach

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Abstract: *Free torsional vibration of doubly symmetric thin-walled beams of open section using spectral dynamics is carried out in this paper. Frequency equation for the case of doubly symmetric thin-walled beam with unequal rotational restraints at the ends of the beam is derived in this paper. The derived Frequency equation with appropriate boundary conditions is derived and is solved for varying values of warping parameter and the rotational restraint parameters. The effects of the unequal rotational restraint parameters and the warping parameter on the free torsional vibration frequencies are investigated in detail. A computer program using MATLAB is developed to solve the spectral frequency equation derived. Natural frequencies for various values of unequal rotational restraint parameters for different values of warping parameter are obtained. Results are presented in both tabular as well as graphical form showing the influence of these parameters on the values of fundamental torsional frequency parameters clearly.*
Keywords: *Beam, Open Section, Torsion, Dynamic Stiffness, Warping*

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I. Introduction

In many practical situations, by using elastically restrained edges against the rotation and translation, one can simulate the complex boundary conditions of structural members.

The problem of vibrations of generally restrained beams with various combinations of boundary conditions has been investigated by many researchers in the available literature Computation of natural frequencies and mode shapes of cantilever beams with flexible roots has been studied well [3, 5, 9-11]. Kameswara Rao and Mirza [12] derived the transcendental frequency equation and mode shape expressions for the case of generally restrained Euler-Bernoulli beams and presented extensive numerical results for various values of linear and rotational restraint parameters.

Strangely, there are quite a good number of publications on flexural vibrations of elastically restrained cantilever beams, the literature on torsional vibrations of doubly symmetric thin-walled beams of open section is surprisingly scarce. Including elastic torsional and warping restraints, Carr [15] and Christino and Salmela [13] presented numerical results using approximate methods for the calculation of natural frequencies. Torsional vibration frequencies for beams of open thin-walled sections, subjected to several combinations of classical boundary conditions were first derived by Gere [14].

Burlon et al [15] proposed an exact approach to coupled bending and torsional free vibration analysis of beams with mono-symmetric cross section, featuring an arbitrary number of in-span elastic supports and attached masses. The proposed method relies on the elementary coupled bending-torsion theory and makes use of the theory of generalized functions to handle the discontinuities of the response variables. In another paper, Burlon et al [16] investigated the stochastic response of a coupled bending–torsion beam, carrying an arbitrary number of supports/masses. Using the theory of generalized functions in conjunction with the Euler–St.Venant coupled bending–torsion beam theory, exact analytical solutions under stationary inputs are obtained based on frequency response functions derived by two different closed-form expressions.

The review presented by Sapountzakis[19], clearly shows that the problem of free torsional vibration analysis of doubly-symmetric thin-walled I-beams or Z-beams subjected to partial warping restraint is not being addressed till now in the available literature. In view of the same, an attempt has been made in this paper to present a spectral dynamic analysis of free torsional vibration of doubly-symmetric thin-walled beams of open section with unequal rotational restraints at the ends of the beam including the effects of non-linear warping parameter.

The resulting spectral frequency equation is solved for varying values of warping parameter and the partial rotational restraint parameters. The influence of unequal rotational restraint parameters along with warping parameter on the free torsional vibration frequencies is investigated in detail by utilizing a computer program developed using MATLAB, to solve the spectral frequency equation derived in this paper. Numerical results for natural frequencies for various values of unequal partial rotational restraint parameters are obtained and presented in both tabular as well as graphical form for use in design, showing their parametric influence clearly.

II. Formulation and Analysis

Consider a long doubly-symmetric thin-walled beam of open cross section of length L and the beam as undergoing free torsional vibrations. The corresponding differential equation of motion can be written as:

$$EC_W \frac{\partial^4 \varphi}{\partial z^4} - GC_S \frac{\partial^2 \varphi}{\partial z^2} + \rho I_P \frac{\partial^2 \varphi}{\partial t^2} = 0 \quad (1)$$

where,

E= young's modulus, C_W=warping constant,G =shear modulus,C_S = torsion constant, ρ =mass density of the material of the beam, I_P =polar moment of inertia, φ = angle of twist, z= distance along the length of the beam.

For free torsional vibrations, the angle of twist $\varphi(z,t)$ can be expressed in the form.

$$\varphi(z,t) = x(z)e^{i\omega t} \quad (2)$$

$$x(z) = Ce^{mz} \quad (3)$$

In which $x(z)$ is the modal shape function corresponding to each beam torsional natural frequency ω.

The expression for $x(z)$which satisfies Eqn. (1) can be written as:

$$x(z) = Ae^{+\alpha z} + Be^{-\alpha z} + Ce^{+i\beta z} + De^{-i\beta z} \quad (4)$$

in which,

$$\beta L, \alpha L = \sqrt{\frac{\mp K^2 + \sqrt{K^4 + 4\lambda^2}}{2}} \quad (5)$$

where,

$K^2 = \left(\frac{GC_S L^2}{EC_W}\right)$; Non- dimensional warping parameter

$\lambda^2 = \left(\frac{\rho I_P \omega^2 L^4}{EC_W}\right)$; Non- dimensional frequency parameter

From Eqn. (4), we have the following relation between αL and βL

$$(\alpha L)^2 = (\beta L)^2 + K^2 \quad (6)$$

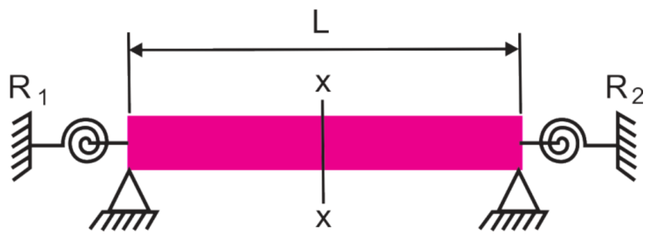
KnowingαL and βL, the frequency parameter λ can be evaluated using the following equation:

$$\lambda^2 = (\alpha L)(\beta L) \quad (7)$$

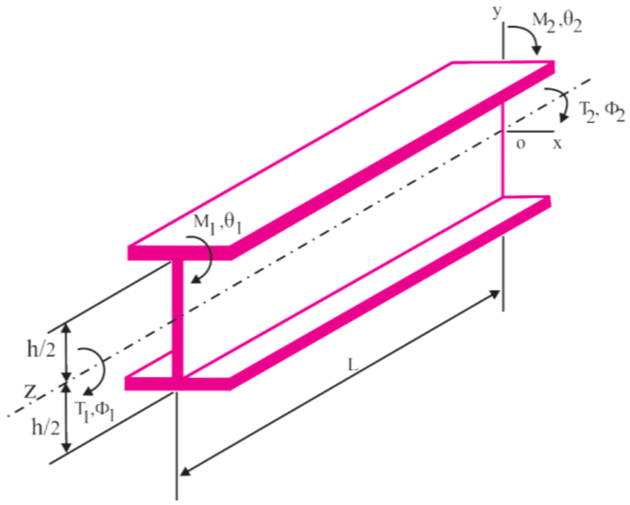
The four arbitrary constants A, B, C and D in Eqn. (4) can be determined from the boundary conditions of the beam. For any single-span beam, there will be two boundary conditions at each end and these four conditions then determine the corresponding frequency expression.

III. Derivation of Spectral Frequency Equation

Consider a thin-walled doubly symmetric I-beam with both the ends restrained by unequal rotational springs as shown in figure 1, undergoing free torsional vibrations. In order to derive the spectral frequency equation for this case, let us first introduce the related nomenclature.



(a)



(b)

Figure 1(a). Thin-Walled Open Section I-Beam with Unequal Rotational Restraints at the Ends. (b). Cross-section of the beam (at x-x).

The variation of angle of twist φ with respect to z is denoted by θ*(z)*. The flange bending moment and the total twisting moment are given by *M(z)* and *T(z)*. Considering clockwise rotations and moments to be positive, we have

$$\theta(z) = \frac{d\varphi}{dZ};\ hM(z) = -EC_W\frac{d^2\varphi}{dz^2} \quad (8)$$

$$T(z) = -EC_W\frac{d^3\varphi}{dz^3} + GC_S\frac{d\varphi}{dZ} \quad (9)$$

where $EC_W = \frac{I_f h^2}{2}$. I_f being the flange moment of inertia and h is the distance between the centerlines of the flanges of a thin-walled I-beam.

Taking S1 and S2 as the stiffnesses of the rotational springssituated at both ends and R_1 = (S_1L/EC_W) and R_2= (S_2L/EC_W) as the non-dimensional rotational spring stiffness parameters and Z=(z/L) as the non-dimensional length of the beam, the boundary conditions can be easily identified as follows:

$$\text{At Z =0, } \varphi = 0, \frac{d^2\varphi}{dZ^2} = R_1\frac{d\varphi}{dZ} \quad (10)$$

$$\text{And at Z = L, } \varphi = 0, \frac{d^2\varphi}{dZ^2} = R_2\frac{d\varphi}{dZ} \quad (11)$$

Applying the above given boundary condition, the spectral frequency equation obtained for this case under study is as given below:

$$R_1R_2(F_2Q_{1m2} + S_1) + (R_1 + R_2)F_1S_2 + F_3Q_{1m2} = 0 \quad (12)$$

where

$$F_2 = \frac{(\alpha^2-\beta^2)}{(\alpha\beta)};\ F_3 = \frac{(\alpha^2+\beta^2)}{(\alpha\beta)};\ F_4 = \frac{(\alpha^4+2\alpha^2\beta^2+\beta^4)}{(\alpha\beta)}; \quad (13)$$

$$Q_1 = \frac{1+e^{2L(\alpha+i\beta)}}{4e^{L(\alpha+i\beta)}};\ Q_2 = \frac{1+e^{2L(\alpha-i\beta)}}{4e^{L(\alpha-i\beta)}};\ Q_3 = \frac{1-e^{2L(\alpha+i\beta)}}{4ie^{L(\alpha+i\beta)}};\ Q_4 = \frac{1-e^{2L(\alpha-i\beta)}}{4ie^{L(\alpha-i\beta)}} \quad (14)$$

$$Q_{1p2} = (Q_1 + Q_2), Q_{1m2} = (Q_1 - Q_2), Q_{3p4} = (Q_3 + Q_4), Q_{3m4} = (Q_3 - Q_4) \quad (15)$$

$$S_1 = 2(1 - Q_{1p2});\ S_2 = (\alpha^3Q_{3p4} - \beta^3Q_{3m4});\ S_3 = (\alpha Q_{3m4} - \beta Q_{3p4}) \quad (16)$$

Three degenerate cases of spectral frequency equations can be easily obtained from Equation (12) as follows:

(i) For $R_1 = 0$ and $R_2 = 0$, we get the case of simply supported beam for which we obtain

$$Q_{1m2} = 0 \quad (17)$$

(ii) For $R_1 = 0$ and $R_2 = \infty$, we get the case ofa beamsimply-supported at one end and clamped at the other end for which we obtain

$$S_2 = 0 \quad (18)$$

(iii) For $R_1 = \infty$ and $R_2 = \infty$, we get the case of a beam clamped at both ends for which we obtain

$$F_2Q_{1m2} + S_1 = 0 \quad (19)$$

IV. Results and Discussions

Numerical results for the first three natural torsional frequencies of vibration of thin-walled beams of open section are obtained by solving the transcendental spectral frequency Eq. (12) using trial-and-error method. The Muller's iteration technique based on bisection method is coded in MATLAB and the same is utilised in generating the numerical results whichare presented in several tables and graphs suitable for use in design.

It should be mentioned here that even though several studies are made by researchers in the area of torsional frequencies of thin-walled beams of open section, numerical values are not made available for use in design. As is known, graphical results can help us only in understanding the trend of variation of natural frequencies due to the increase in warping parameter K and the partial warping restraint parameters $R1$ and $R2$, but will not provide the frequencies to the four digit accuracy which we require for using the same for design.

For the case of unequally rotationally restrained thin-walled beam with partially restrained warping ($R1$) varying from o to 10^{+18} at the left end and with partial warping restraint (R2) varying from 0 to 10^{+18} at the other end, the fundamental mode torsional frequencies for a fixed value of warping parameter K=0.0 are presented in Table 1. The fundamental mode torsional frequencies are determined for a wide range of $R1$ and R2 but only a representativeset of values are presented in Table 1. Figure 2 represents the variation of frequency parameter with rotational restraints ($R1$& $R2 = 0$ to 10^{+18}) for a givenwarping parameter, $K = 0$, whereas, Figure 2 (a) is drawn to clearly show the variation of the fundamental first mode frequencies with different values of $R1$ and $R2$. Similarly, the fundamental frequency parameters with warping parameter ($K = 0\ to\ 500$) for $R1$and $R2$ varying from 0 to 10^{+18} are computed and presented in Tables 2 and 3. Also, the variation of frequency parameter with warping parameter ($K = 0\ to\ 500$) for $R1$ and $R2$ varying from 0 to 10^{+18} are shown in Figure 3 to 8.

Table 1. First mode natural frequencies for various values of rotational restraint parameters R1 and R2 and for warping parameter $K = 0.0$.

$R1$	$R2 = 0$	$R2 = 0.01$	$R2 = 0.1$	$R2 = 1$	$R2 = 10$	$R2 = 100$	$R2 = 1000$	$R2 = 10^{18}$
0	3.1416	3.1432	3.1572	3.2733	3.6646	3.8892	3.9227	3.9266
0.01	3.1432	3.1448	3.1588	3.2748	3.6660	3.8905	3.9240	3.9279
0.1	3.1572	3.1588	3.1727	3.2881	3.6781	3.9024	3.9359	3.9398
1	3.2733	3.2748	3.2881	3.3988	3.7806	4.0043	4.0379	4.0418
10	3.6646	3.6660	3.6781	3.7806	4.1557	4.3900	4.4260	4.4303
100	3.8892	3.8905	3.9024	4.0043	4.3900	4.6413	4.6807	4.6853
1000	3.9227	3.9240	3.9359	4.0379	4.4260	4.6807	4.7206	4.7253
10^{18}	3.9266	3.9279	3.9398	4.0418	4.4303	4.6853	4.7253	4.7300

Table 2. First mode natural frequencies for various values of rotational restraint parameters R1 and R2 and for warping parameter $K = 1.0$.

$R1$	$R2 = 0$	$R = 0.01$	$R2 = 0.1$	$R2 = 1$	$R2 = 10$	$R2 = 100$	$R2 = 1000$	$R2 = 10^{18}$
0	3.2183	3.2198	3.2328	3.3416	3.7165	3.9363	3.9694	3.9733
0.01	3.2198	3.2213	3.2343	3.3430	3.7178	3.9376	3.9707	3.9746
0.1	3.2328	3.2343	3.2473	3.3555	3.7293	3.9490	3.9821	3.9860
1	3.3416	3.3430	3.3555	3.4601	3.8275	4.0471	4.0803	4.0842
10	3.7165	3.7178	3.7293	3.8275	4.1916	4.4227	4.4584	4.4626
100	3.9363	3.9376	3.9490	4.0471	4.4227	4.6707	4.7098	4.7144
1000	3.9694	3.9707	3.9821	4.0803	4.4584	4.7098	4.7495	4.7541
10^{18}	3.9733	3.9746	3.9860	4.0842	4.4626	4.7144	4.7541	4.7588

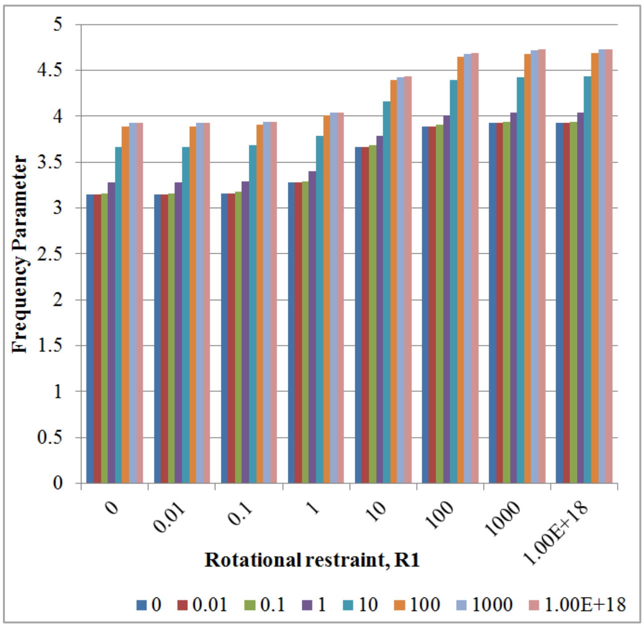
Table 3. First mode natural frequencies for various values of rotational restraint parameters R1 and R2 and for warping parameter $K = 500$.

$R1$	$R2 = 0$	$R = 0.01$	$R2 = 0.1$	$R2 = 1$	$R2 = 10$	$R2 = 100$	$R2 = 1000$	$R2 = 10^{18}$
0	39.6337	39.6337	39.6337	39.6337	39.6344	39.6403	39.6601	39.6734
0.01	39.6337	39.6337	39.6337	39.6337	39.6344	39.6403	39.6601	39.6734
0.1	39.6337	39.6337	39.6337	39.6338	39.6344	39.6403	39.6601	39.6734
1	39.6337	39.6337	39.6344	39.6338	39.6345	39.6404	39.6602	39.6734
10	39.6344	39.6344	39.6344	39.6345	39.6352	39.6410	39.6609	39.6741
100	39.6403	39.6403	39.6403	39.6404	39.6410	39.6469	39.6667	39.6800
1000	39.6601	39.6601	39.6601	39.6602	39.6609	39.6667	39.6866	39.6999
10^{18}	39.6734	39.6734	39.6734	39.6734	39.6741	39.6800	39.6999	39.7132

It has been observed from Table 1 to 3 that the increase in warping parameter K is to increase the fundamental mode torsional frequencies significantly. However, the change is very marginal when warping parameter is very low. But, the change is phenomenal at higher values of warping parameter K. For values of K greater than 10, we can easily notice that the frequencies of un-symmetrically rotationally restrained beam almost tend to converge to a constant value as K approaches higher values (refer Table 4). For a constant value of warping parameter K, the increasing the values of partial warping parameters $R1$ and $R2$ from 0 to infinity (10^{18}) results in consistent increase in the values of fundamental mode frequencies as the un-symmetrically rotationally restrained ends become stiffer and stiffer.

Table 4. The percentage variation of frequency parameter with increasing values of $R1$ and K and as $R2$ varies from 0 to 10e+18.

	K=0.01	K=0.1	K=1	K=10	K=100	K=500
R1=0	24.98727	24.98727	24.97136	5.40046	0.503697	0.100167
R1=0.01	24.965	24.96819	24.94911	5.400178	0.503697	0.100167
R1=0.1	24.78779	24.78779	24.77201	5.40324	0.503695	0.100167
R1=1	23.47784	23.47784	23.46671	5.428105	0.503672	0.100167
R1=10	20.8945	20.8945	20.8829	5.586165	0.504031	0.100166
R1=100	20.46951	20.46951	20.46173	5.80139	0.506375	0.100151
R1=1000	20.4604	20.4604	20.45269	5.851407	0.508139	0.100353
R1=10e+18	20.46045	20.46045	20.45275	5.857844	0.50903	0.100319



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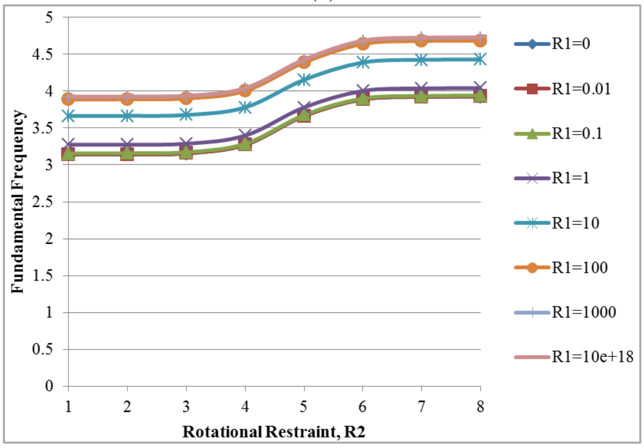

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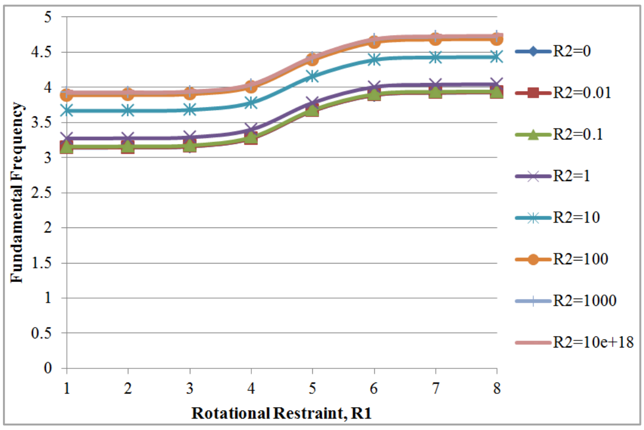


(c)

Figure 2. (a), (b) and (c). Variation of frequency parameter with rotational restraints (R1&R2=0 to 10^{18}) for a given Warping parameter (K=0).

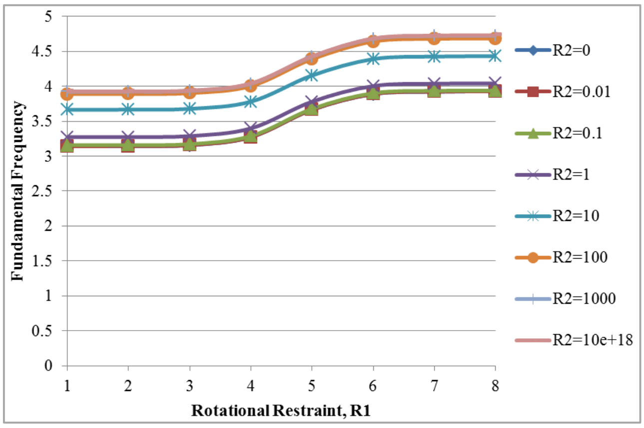


Figure 3. Variation of frequency parameter with rotational restraints (R1&R2=0 to 10^{18}) for a given Warping parameter (K=0.01).

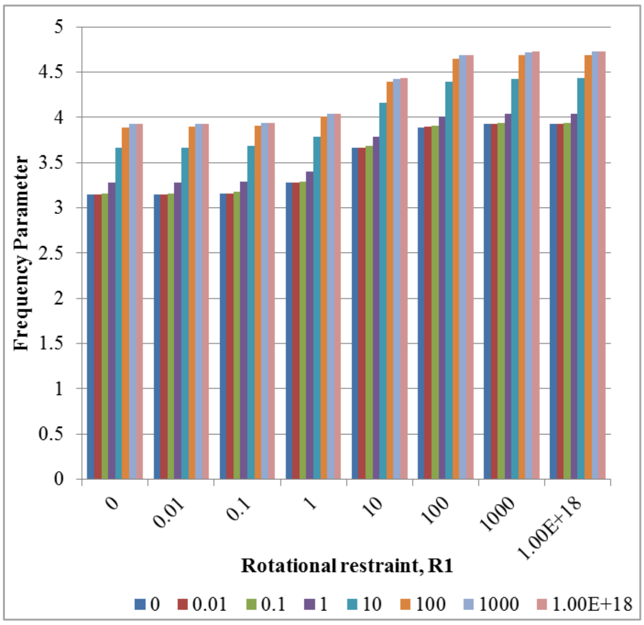


Figure 4. Variation of frequency parameter with rotational restraints (R1&R2=0 to 10^{18}) for a given Warping parameter (K=0.1).

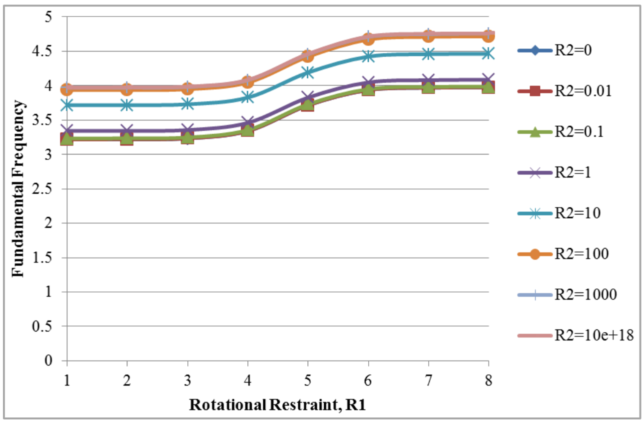
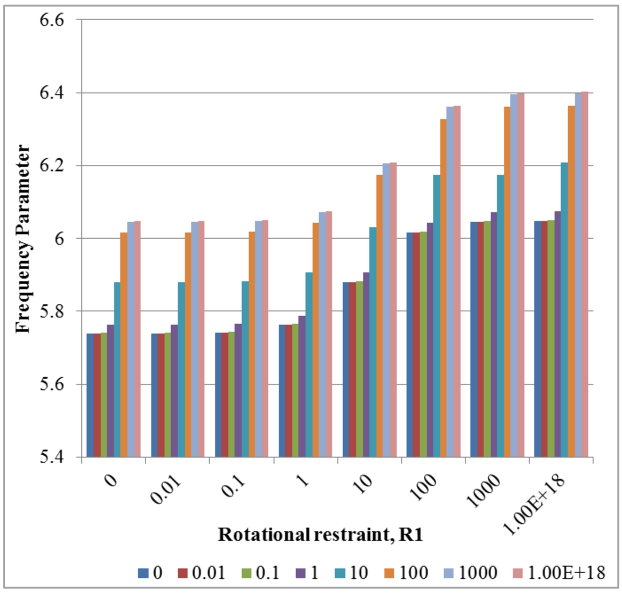
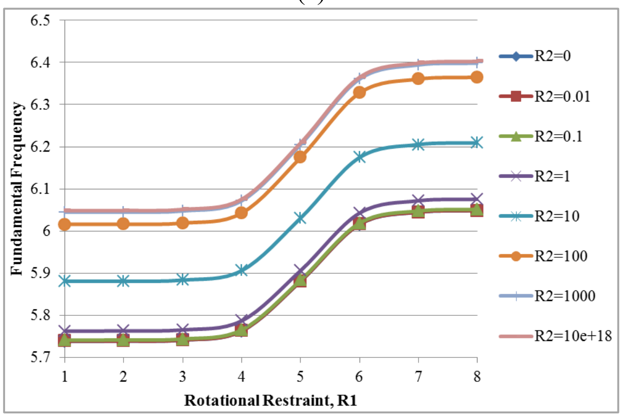


Figure 5. Variation of frequency parameter with rotational restraints (R1&R2=0 to 10^{18}) for a given Warping parameter (K=1).



(a)



(b)

Figure 6. (a) and (b). Variation of frequency parameter with rotational restraints (R1&R2=0 to 10^{18}) for a given Warping parameter (K=10).

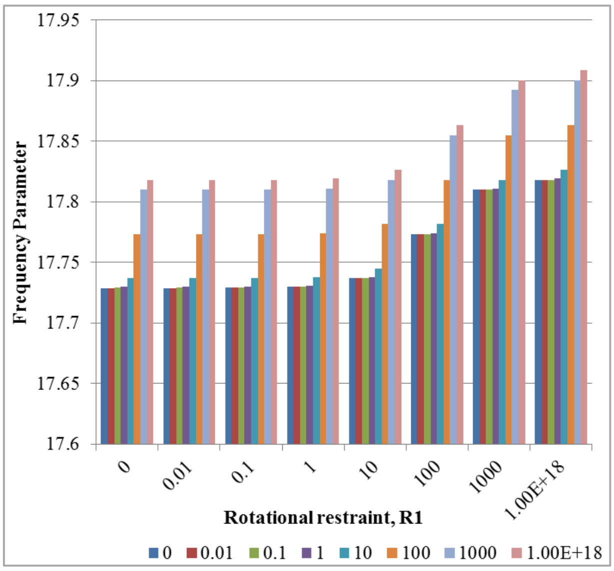


Figure 7. Variation of frequency parameter with rotational restraints (R1&R2=0 to 10^{18}) for a given Warping parameter (K=100).

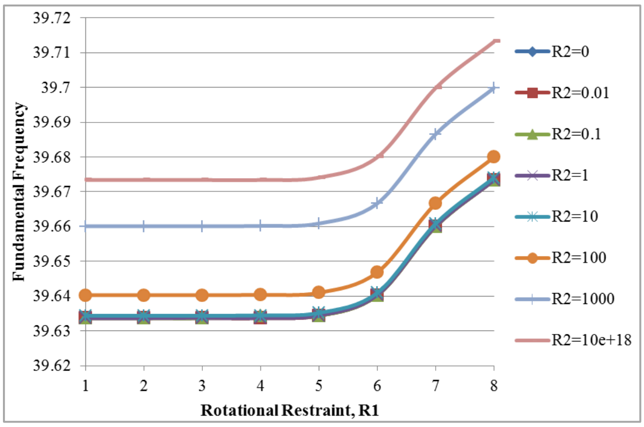


Figure 8. Variation of frequency parameter with rotational restraints (R1&R2=0 to 10^{18}) for a given Warping parameter (K=500).

From the definition of non-dimensional warping parameter K, we can understand that the torsional frequency increases for increasing values of torsion constant C_S or for decreasing values of warping constant C_W. Effect of K also can be seen to be more predominant compared to the effect of partial warping restraint R. This can be observed from Figure 2 to 8, whereaswhen$K$ is increasing from 0 to 500, the two curves related to the un-symmetrically rotationally restrained beam with full restraint and the one with unrestrained warping are almost converging to the same value and hence we can conclude that the boundary condition has insignificant effect on the natural torsional frequencies of thin-walled doubly symmetric beams for very high values of warping parameter$K$.

Fundamental mode torsional frequencies of thin-walled beams for a wide range of values of warping parameter K from 0.01 to 500 and the partial warping restraints $R1$ and R2 from 0 to 10^{18}arecalculated. These results are also plotted in Figures 2 to 8 showing clearly the variation of fundamental natural torsional frequency with varying values of warping parameter K and the partial warping restraint parameters R1 and R2. The percentage variation of frequency parameter with increasing values of $R1$ and K and as $R2$ varies from 0 to $10^{+18}$is presented in Table 4. The percentage variation of frequency parameter changes from 24.98 to 20.46for a given value of K and as R1 and R2 varies from 0 to 10^{+18}. The change of percentage decreases with K for a given value of $R1$ and as $R2$ vary from 0.01 to 10^{+18} as shown in Table 4.

By using latest approximate methods such as Generalised Differential Quadrature Method (GDQM), Differential Transform Method (DTM), Adomian Decomposition Method (ADM). or any other method such as Finite Element Method one can attempt to obtain results and compare the same with the results presented in this paper and determine their accuracy.

V. Concluding Remarks

For the case of an un-symmetrically rotationally restrained thin-walled beam of doubly symmetric open cross-section undergoing free torsional vibrations and subjected to partial warping restraint, the spectral frequency equation is derived in this paper. The resulting transcendental frequency equation for the case of un-symmetrically rotationally restrained boundary conditions is solved for thin-walled beams of open cross section for varying values of warping parameter and the partial warping restraint parameters. Using a MATLAB computer program developed to solve the spectral frequency equation derived.

The influence of partial warping restraint parameter R1 and R2 and the warping parameter Kon the free torsional vibration frequencies is investigated in detail and significant amount of numerical frequency data is generated. The results obtained are presented in both tabular as well as graphical form showing their parametric influence clearly. In comparison with the partial warping restraint parameter R1 and R2 , the warping parameter K is found to have significant effect on the torsional natural frequencies. Spectral dynamic analysis of free torsional vibration of doubly symmetric thin-walled beams of open section is carried out and detailed results of this study are presented in this paper suitable for use in design and also for checking approximate solutions obtained for their accuracy.

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