Integral solutions of Quadratic Diophantine equation
\[10w^2 - x^2 - y^2 + z^2 = t^2\] with five unknowns

R.Anbuselvi¹, * S.Jamuna Rani²

¹Associate Professor, Department of Mathematics, ADM College for women, Nagapattinam, Tamilnadu, India
²Asst Professor, Department of Computer Applications, Bharathiyar college of Engineering and Technology, Karaikal, Puducherry, India
Corresponding Author: R.Anbuselvi

ABSTRACT: The quadratic Diophantine equation given by \(10w^2 - x^2 - y^2 + z^2 = t^2\) is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.

Keywords: Ternary quadratic, integral solutions, polygonal numbers.

I. INTRODUCTION

Quadratic equations are rich in variety [1-3]. For an extensive review of sizable literature and various problems, one may refer [1-16]. In this communication, we consider yet another interesting quadratic equation \(10w^2 - x^2 - y^2 + z^2 = t^2\) and obtain infinitely many non-trivial integral solutions. A few interesting relations between the solutions and special Polygonal numbers are presented.

Notations Used
- \(t_{m,n}\) - Polygonal number of rank ‘n’ with size ‘m’
- \(CP_n^6\) - Centered hexagonal Pyramidal number of rank n
- \(Gn_n\) - Gnomic number of rank ‘n’
- \(FN_n^4\) - Figurative number of rank ‘n’ with size ‘m’
- \(Pr_n\) - Pronic number of rank ‘n’
- \(P_n^m\) - Pyramidal number of rank ‘n’ with size ‘m’
- \(kyn\) - Keynea number

II. METHODS OF ANALYSIS

The Quadratic Diophantine equation with five unknowns to be solved for its non zero distinct integral solutions is

\[10w^2 - x^2 - y^2 + z^2 = t^2\] (1)

On substituting the linear transformation
\[
\begin{align*}
x &= w + z \\
y &= w - z
\end{align*}
\]

in (1), it leads to

\[8w^2 - t^2 = z^2\] (3)

We obtain different patterns of integral solutions to (1) through solving (3) which are illustrated as follows:

Pattern 1

Equation (3) can be written as

\[
\frac{2w + t}{z + 2w} = \frac{A}{B}, B \neq 0
\]

Considering

\[
\frac{z - 2w}{2w - t} = \frac{A}{B}
\]
Integral Solutions Of Quadratic Diophantine Equation

Which is equivalent to the system of equations

\[
\begin{align*}
    w &= w(A, B) = A^2 + B^2 \\
    t &= t(A, B) = 2A^2 + 4AB - 2B^2 \\
    z &= z(A, B) = 2B^2 + 4AB - 2A^2
\end{align*}
\]  

Substituting (4) and (3) in (2), the corresponding non-zero distinct integral solutions of (1) are given by

\[
\begin{align*}
    x(A, B) &= x = 3B^2 - A^2 + 4AB \\
    y(A, B) &= y = 3A^2 - B^2 - 4AB \\
    z(A, B) &= z = 2B^2 + 4AB - 2A^2 \\
    w(A, B) &= w = A^2 + B^2 \\
    t(A, B) &= t = 2A^2 + 4AB - 2B^2
\end{align*}
\]  

**Properties**

1. \( x(A - 1) + 3y(A - 1) - 16 t_{3,A} \equiv 0 \)
2. \( 2x(1, B) + t(1, B) - 8 t_{3, A} \equiv 0 \ (mod \ 8) \)
3. \( t(A, 1) + 2w(A, 1) - 8 t_{3, A} \equiv 0 \)
4. \( t(A(A + 2), 1) + 2w(A(A + 2), 1) - 24 P_{A}^2 \equiv 0 \)
5. \( t(A(A + 2)(A + 3), 1) + 2w(A(A + 2)(A + 3), 1) - 96 P_{A}^4 \equiv 0 \)
6. \( t(A(A + 1)(A + 2), 1) + 2w(A(A + 1)(A + 2), 1) - 48 F_{4,n - 4} \equiv 0 \)
7. \( z(1, B) - 4 t_{3, B} \equiv -2 (mod \ 2) \)
8. \( 2x(1, B) + t(1, B) - t_{8, B} = 0 (mod \ 12) \)
9. \( x(A, 1) + y(A, 1) - t_{4, A} \equiv 2 (mod \ 3) \)
10. \( x(A, A) + y(A, A) \) can be expressed as a perfect square.

11. Each of the following expressions represents a perfect number
   (i) \([t(2, 1) + x(1, 2)]\)
   (ii) \([x(2, 1) + x(2, 2)]t(2, 2)\)
   (iii) \(x(1, 1)\)
   (iv) \(t(1, 2) + y(1, 1)\)
   (v) \(x(2, 2) + x(1, 1)\)

12. Each of the following expressions represents a perfect square
   (i) \(w(3, 3) + x(2, 1)\)
   (ii) \(x(3, 2) + x(2, 2) + y(2, 1)\)
   (iii) \(2x(3, 3) + x(1, 2) + y(2, 1)\)

13. Each of the following expressions represents a cube number
   (i) \(x(3, 3) + x(1, 2)\)
   (ii) \(t(1, 2)\)
   (iii) \(z(1, 2) + w(1, 2) + t(1, 2)\)
   (iv) \(x(3, 3) + 2z(3, 3) - \frac{1}{2} w(1, 1)\)
   (v) \(4x(3, 3)\)

14. Each of the following expressions represents nasty number
   (i) \(x(2, 2) + x(1, 1)\)
   (ii) \(x(1, 1)x(2, 2)\)
   (iii) \(x(1, 1)w(1, 2)\)
   (iv) \(y(2, 1)x(2, 1)w(2, 1)\)
   (v) \(x(2, 2)w(1, 2)\)

**Pattern II**

The equation (3) can be written as

\[8w^2 - t^2 = z^2 * 1\]  \(\text{(6)}\)

Then write

\[
\begin{align*}
    z &= 8a^2 - b^2 \\
    1 &= (\sqrt{k} + 1)(\sqrt{k} - 1)
\end{align*}
\]  \(\text{(7)}\)

Substituting (7) into (6) reduces to
\[8w^2 - t^2 = (2\sqrt{2}w + t)(2\sqrt{2}w - t)\]

Equating rational and irrational parts, the coefficient values are

\[
\begin{align*}
w &= w(a, b) = 4a^2 + \frac{b^2}{2} + 2ab \\
t &= t(a, b) = 8a^2 + b^2 + 8ab \\
z &= z(a, b) = 2b^2 + 4ab - 2a^2
\end{align*}
\]  

\[\tag{8}\]

As our interest is on finding integer solutions, it is seen that the values of \(x, y\) and \(z\) are integers when both \(a\) and \(b\) are of the same parity. Thus by taking \(a = 2A, \quad b = 2B\in (7)\) and substituting the corresponding values of \(u, v\) in (2) the non-zero integral solutions of (1) are given by

\[
\begin{align*}
x &= x(A,B) = 12A^2 - 2B^2 + 4AB \\
y &= y(A,B) = -4A^2 + 6B^2 + 4AB \\
z &= z(A,B) = 8A^2 - 4B^2 \\
w &= w(A,B) = 4A^2 + 2B^2 + 4AB \\
t &= t(A,B) = 8A^2 + 4B^2 + 16AB
\end{align*}
\]  

\[\tag{9}\]

Properties

1. \(w(A,B) - y(A,B) = z(A,B)\)
2. \(z(A,1) + t(A,1) - 16\) if \(A\) is odd
3. \(w(A,1) - y(A,1) - 8\) if \(A\) is even
4. \(x(1,B) + w(1,B) - 4Gn_0 - 20\) if \(B\) is odd
5. \(2y(1,B) + z(1,B) - 16t_B\) if \(B\) is even
6. \(w(A,1) + t(A,1) - 24t_B\) if \(B\) is odd
7. \(x(1,B) + 3y(1,B) - 32t_B = 0\) if \(B\) is even
8. \(x(1,B(B+2)) + 3y(1,B(B+2)) - 96 P_{A1} = 0\)
9. \(x(1,B^2(B+1)) + 3y(1,B^2(B+1)) - 32 P_{A3} = 0\)
10. \(x(1,B(B+2)(B+3)) + 3y(1,B(B+2)(B+3)) - 384 P_{A4} = 0\)

11. Each of the following expressions represents a Nasty number
   (i) \([t(1,1) + x(1,1)]\)
   (ii) \([t(2,3) + x(2,2)]\)
   (iii) \(x(3,3) - x(1,3)\)
   (iv) \(x(3,1) + w(1,2) + x(1,0)\)
   (v) \(w(1,1) + x(1,0) + z(1,0)\)
   (vi) \(t(1,1) - z(1,1)\)

12. Each of the following expressions represents a perfect number
   (i) \(t(3,3) + 2x(3,2) - z(1,1)\)
   (ii) \(4t(3,1)\)
   (iii) \(8[t(3,3) + t(2,3) + t(3,2) + t(3,1) + x(3,1) + x(3,3)] + z(2,2)y(2,2)\)
   (iv) \(t(1,2) - w(1,2)\)
   (v) \(z(2,2) - w(1,1)\)

13. Each of the following expressions represents a cube number
   (i) \(t(1,3) + t(3,1)\)
   (ii) \(t(3,3) + t(3,2) + z(3,2) + w(1,2)\)
   (iii) \(4t(3,3) - z(1,0)\)
   (iv) \(t(3,3) - z(3,3)\)
   (v) \(z(3,1) - w(1,0)\)

14. Each of the following expressions represents perfect square
   (i) \(w(2,2) + y(2,2)\)
   (ii) \(t(3,3) + t(0,1)\)
   (iii) \(x(3,1) + z(2,2) + w(1,1)\)
   (iv) \(2t(3,2) + t(2,3) + t(3,3)\)
   (v) \(y(2,3) + w(0,1)\)

**Pattern III**

Consider the linear transformation

\[\]
\[ w = u - v, \quad t = 2u - 4v \]  

(10)

Substituting (9) into (3) the equation reduces to

\[ \frac{2u + z}{8v} = \frac{v}{2u - z} = \frac{A}{B}, \quad B \neq 0 \]

Simplifying and employing the method of factorization,

\[ u = 8A^2 + B^2 \]
\[ v = 4AB \]
\[ z = 3A^2 + B^2 \]

Substituting the corresponding values of \( u, v \) in (2) the non-zero integral solutions of (1) are given by

\[
\begin{align*}
    x(A,B) &= x = 24A^2 - B^2 - 4AB \\
    y(A,B) &= y = 3B^2 - 8A^2 - 4AB \\
    z(A,B) &= z = 16A^2 - 2B^2 \\
    w(A,B) &= w = 8A^2 + B^2 - 4AB \\
    t(A,B) &= t = 16A^2 + 2B^2 - 16AB
\end{align*}
\]

(11)

**Properties**

1. \( z(A,1) - t(A,1) \equiv -4 (mod 16) \)
2. \( y(1,B) + w(1,B) - 4 \text{Pr}_A \equiv 0 (mod 12) \)
3. \( 3w(1,B) - y(1,B) + 9Gn_0 + 41 \equiv 0 \)
4. \( y(A,A^2) + w(A,A^2) + 8CP_n - 4t_{4,A} \equiv 0 \)
5. \( x(B,B) - w(B,B) - 14t_{4,B} \equiv 0 \)
6. Each of the following expressions represents a Nasty number
   (i) \( [2x(1,2)] \)
   (ii) \( [t(2,1) - w(1,2)] \)
   (iii) \( x(2,3) - x(1,3) \)
   (iv) \( x(3,2) - z(1,2) \)
   (v) \( x(3,3) - 3y(1,3) \)
7. Each of the following expressions represents a perfect number
   (i) \( w(2,2) + t(2,2) \)
   (ii) \( z(3,1) + x(3,2) + x(3,1) + y(2,1) \)
   (iii) \( 2x(3,1) + x(3,2) + w(2,2) + y(1,3) \)
   (iv) \( y(1,2) \)
   (v) \( z(1,1) - z(1,2) \)
8. Each of the following expressions represents a cube number
   (i) \( x(3,2) - y(1,2) \)
   (ii) \( z(2,2) + t(2,2) \)
   (iii) \( x(3,1) + z(3,1) \)
   (iv) \( 2x(3,1) + x(3,1) + w(2,2) + w(1,1) \)
   (v) \( 3z(3,1) + 2[x(3,1) - y(3,1) + w(1,2)] \)
9. Each of the following expressions represents perfect square
   (i) \( x(2,1) - y(2,1) + w(2,2) \)
   (ii) \( x(3,2) + z(3,2) \)
   (iii) \( x(3,1) - y(1,1) + x(3,2) \)
   (iv) \( z(3,3) - w(1,1) \)
   (v) \( t(3,3) + y(1,3) \)

**Pattern IV**

Consider the transformation

\[
(2u + z)(2u - z) = 8 \cdot v^2
\]

(12)

Equating positive and negative factor, we get

\[
(2u + z) = v^2 \quad \text{(12a)}
\]
\[
(2u - z) = 8 \quad \text{(12b)}
\]

Solving the above equations, we get
Integral Solutions Of Quadratic Diophantine Equation

\[
\begin{align*}
u &= \frac{v^2 + 8}{4} \\
z &= \frac{v^2 - 8}{2}
\end{align*}
\]  \hspace{1cm} (13)

As our interest on finding integer solutions, choose \(v\) so that \(u\) and \(z\) are integers
Then write
\[
\begin{align*}
u &= A^2 + 2A + 3 \\
v &= 2A + 2
\end{align*}
\]  \hspace{1cm} (14)

Substituting (14) in (13), the solutions are
\[
\begin{align*}
x(A, B) &= x = 3A^2 + 4A - 1 \\
y(A, B) &= y = -A^2 - 4A + 3 \\
z(A, B) &= z = 2A^2 + 4A - 2 \\
w(A, B) &= w = A^2 + 1 \\
t(A, B) &= t = 2A^2 - 4A - 2
\end{align*}
\]  \hspace{1cm} (15)

Properties

1. \(w(2) - y(7)\) is a Kynea number
2. \(y(A) - x(A) + 2z(A) = 0\)
3. \(y(A^2) + w(A^2) + 4CP_5 \equiv 4\)
4. \(z(A) + 2w(A) - 2Gn_0 \equiv 2\)
5. Each of the following expression represents a perfect Square
   (i) \(z(9)\)
   (ii) \(x(7) + y(7)\)
   (iii) \(z(9) + t(9)\)
   (iv) \(x(9) + z(9) + 2t(3)\)
   (v) \(x(8) + x(10) + t(7) - t(3)\)
6. Each of the following expression represents a perfect number
   (i) \([x(9) + x(8) - t(3)]\)
   (ii) \(z(3)\)
   (iii) \(2[x(8)] + w(7)\)
   (iv) \(z(7) + y(7) = w(7)\)
7. Each of the following expression represents a cube number
   (i) \(x(10) + z(1)\)
   (ii) \(x(10) + z(10) - w(8)\)
   (iii) \(x(9) + w(8)\)
   (iv) \(w(6) + z(4) - y(5)\)
   (v) \(x(8) + x(10) - w(7)\)
8. Each of the following expression represents a nasty number
   (i) \([w(8) - w(2)]\)
   (ii) \([z(7) - 2w(1)]\)
   (iii) \(w(10) - w(2)\)
   (iv) \(x(3) + z(4) + x(5) + w(1)\)
   (v) \(x(7) - 6z(1)\)

III. CONCLUSION

To conclude, one may search for other patterns of solutions and their corresponding properties.

REFERENCES

Journal Articles

Integral Solutions Of Quadratic Diophantine Equation

[3]. Gopalan M.A, Geetha D. Lattice points on the Hyperboloid of two sheets
\[ x^2 - 6xy + y^2 + 6x - 2y + 5 = z^2 + 4 \]

[4]. Gopalan M.A, Vidyalakshmi S, Kavitha A. Integral points on the Homogenous Cone
\[ z^2 = 2x^2 - 7y^2 \]

[5]. Gopalan M.A, Vidyalakshmi S, Sumathi G. Lattice points on the Hyperboloid of one sheet
\[ 4z^2 = 2x^2 + 3y^2 - 4 \]

\[ 3y^2 = 7x^2 - z^2 + 21 \]

[7]. Gopalan M.A, Vidyalakshmi S, Mallika S. Observation on Hyperboloid of one sheet
\[ x^2 + 2y^2 - z^2 = 2 \]

\[ 6z^2 + 3y^2 - 2x^2 = 0 \]

[9]. Gopalan M.A, Vidyalakshmi S, Lakshmi K. Lattice points on the Elliptic Paraboloid,
\[ 16y^2 + 9z^2 = 4x^2 \]
Bessel J.Math, 2013, 3(2), 137-145.

\[ x^2 + y^2 + xy = 12z^3 \]


\[ x^2 + 4y^2 = 37z^2 \]

\[ 3(x^2 + y^2) - 5xy = 47z^2 \]

[14]. Gopalan M.A, Vidyalakshmi S, Nivetha S. On Ternary Quadratic Equation
\[ 4(x^2 + y^2) - 7xy = 31z^2 \]

[15]. Gopalan M.A, Vidyalakshmi S, Shanthi J. Lattice points on the Homogenous Cone
\[ 8(x^2 + y^2) - 15xy = 56z^2 \]

\[ 8(x^2 + y^2) + 8(x + y) + 4 = 25z^2 \]

[17]. Anbuselvi R, Jamuna Rani S. Integral solutions of Ternary Quadratic Diophantine Equation
\[ 11x^2 - 3y^2 = 8z^2 \]

Reference Books
[20]. Carmichael RD. The Theory of numbers and Diophantine Analysis,