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Discrete Time Batch Arrival Queue with Multiple Vacations

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ABSTRACT:- In this paper we consider a discrete time batch arrival queueing system with multiple vacations. It is assume that the service of customers arrived in the system between a fixed intervals of time T after which the service goes on vacations after completion of one service of cycle is taken up at the boundaries of the fixed duration of time. This is the case of late arrival. In case of early arrival i.e. arrival before the start of next cycles of service. If the customer finds the system empty, it is served immediately. We prove the Stochastic decomposition property for queue length and waiting time distribution for both the models.
Keywords: - Discrete time system, Stochastic decomposition property, Multiple vacations..

I. INTRODUCTION

This Queueing system with server vacation have been analyzed by many researchers to evaluate various performance measures to check the stability of the system. Discrete time models have wide applications to the computer communication networks, transmission line, fixed cycle traffic light problems, ATM, local area computer networks etc. Some important works on discrete time and vacation model are presented as given below. Fuhrmann and Cooper (1985) studied Stochastic decomposition property for M/G/1 queueing system with generalized vacation. Doshi (1986) gave an important survey on queuing systems with vacation. Keilson and Servi (1986) investigated oscillating random walk model for GI/G/1 vacation system with Bernoulli schedules. Gravey et al. (1990) analyzed discrete time $Geom/D/1$ and $Geom/D/1/N$ queueing systems and derived the mean system size and waiting time. Lee (1991) obtained steady state probabilities for batch arrival queueing system with server vacation under control operating policy. Cooper et al. (1992) provided decomposition theorems for polling models in which the switchover times are effectively additive. Takagi (1992) analyzed $M/G/1/N$ queue with multiple server vacations and its applications to a polling model. Takine and Hasegawa (1992) investigated a generalization of the decomposition property in the $M/G/1$ queueing system with server vacations.

Discrete time systems have been described for communication network system. ATM etc. by Takagi (1993) and Bruneel and Kim (1993), briefly. Tsuchiya and Takahashi (1993) analyzed discrete time single server queues with Markov modulated batch Bernoulli input and the finite capacity, Lee et al. (1994) presented an analysis of the $M^x/G/1$ queue with N-policy and multiple vacation. Chaudhry and Zhao (1994) studied $Geom\ (n)/\ Geom(n)/1/N$ discrete time Markovian queue and determined first passage time and busy period distribution for the system. Chaudhury and Gupta (1996) gave an analysis for performance measures of the discrete time $GI/Geom/1/N$ queue.

Altman et al. (2000) approached optimal open-loop control of vacations, polling and service assignment. Dror and Yechiali (2000) studied close polling models with failing nodes. They obtained explicit formulae for the mean number of jobs as well as for the expected cycle duration and system utilization. Shomrony and Yechiali (2001) discussed burst arrival queues with server vacations and random times. Zhang and Tian (2001) discussed discrete time $Geo/G/1$ queue with multiple adaptive vacations and proved the Stochastic decomposition property for some performance measures.

In this paper , we investigate a discrete time batch arrival queueing system with multiple vacations . The fixed time interval T is assumed between successive arrivals which is distributed geometrically with probability λ. We study late arrival models and early arrival models both. In the late arrival model, the customer arrives in the system towards the end of the fixed duration of services time and in the early model, the customer, if finds the system empty, is served immediately. The server goes on vacations after completion of each service cycle, known as exhaustive service and multiple vacation model. The performance measures such as queue length distribution and waiting time distribution are calculated by taking the probability generating function for both the models. We also discuss the Stochastic decomposition property for both the models for the performance measures.

II. NOTATIONS

T Time between successive arrivals which is distributed geometrically with probability λ

$P_r\{T = k\} = \lambda(1-\lambda)^{k-1}$; $k = 1,2,3,\ldots\ldots\ldots\ldots$

S Service Time

$P_r\{S = k\} = g_k$; $k = 1,2,3,\ldots\ldots\ldots\ldots$

R Random variable which represent maximum number of vacations

$P_r\{R = k\} = r_k$; $k = 1,2,3,\ldots\ldots\ldots\ldots$

V Random variable which represent length of vacation

$P_r\{V = k\} = v_k$; $k = 1,2,3,\ldots\ldots\ldots\ldots$

$V(z)$ Probability generating function of V

$G(z)$ Probability generating function of S

N_n The number of customer arriving during $n-th$ service cycle at time T

F_i The number of customer arriving, at least one, just before completion of service of the $i-th$ customer in the system

III. LATE ARRIVAL MODEL

Let us assume that the customers arriving process in the system, follows embedded Markov chain defined by $\{q_k : k \geq 1\}$, just prior to the end of the fixed duration of time with probability λ.

Following (Zhang and Tian (2001)) and Takagi (1993), we define embedded Markov chain as

$$q_{n+1} = \begin{cases} q_n - 1 + A & ; \quad q_k > 0 \\ q & ; \quad q_k = 0 \end{cases} \tag{1}$$

where A denotes the number of customers arriving during the service of a customer in the system and q be the number of customer left in the queue during the first customer service. So

$$P_r[A = k] = a_k = \sum_{i=1}^{\infty} g_k \binom{k}{i} \lambda^{i-k} (1-\lambda)^{i-j} C_k \quad ; \quad k = 0,1,2,\ldots\ldots\ldots \tag{2}$$

and

$$P_r[x = k] = R[V(1-\lambda)]a_k + \frac{[1-R[V(1-\lambda)]]}{[1-V(1-\lambda)]} \sum_{i=1}^{\infty} g_k\, a_{k+1-i}\, C_k \quad ; \ k = 0,1,2,\ldots\ldots\ldots \tag{3}$$

3.1Queue Length Distribution

Define steady state distribution which exists for Markov chain $\{L_n ; \quad n = 1,2,3,\ldots\ldots.\}$ and it is denoted by

$$\pi_k = \lim_{n\to\infty} P_r\{L_n = k\} \qquad ; k = 0,1,2,3,\ldots\ldots\ldots\ldots\ldots$$

$$\pi(z) \sum_{k=0}^{\infty} \pi_k z^k \tag{4}$$

Using (Zhang and Tian (2001)), we have

$$\pi_j = \pi_0 b_j + \sum_{k=1}^{j+1} \pi_k a_{j+1-k} C_k \qquad ; \quad j \geq 1 \tag{5}$$

From equation (4) and (5), we have the following generating function.

$$Q(z) = \pi_0 B[X(z)] + \frac{1}{z}(Q(z) - \pi_0)[A[X(z)]] \tag{6}$$

where,

$$A[X(z)] = \sum_{k=0}^{\infty} C_k a_k z^k = G[1 - \lambda(1 - X(z))] \tag{7}$$

and

$$B[X(z)] = \sum_{k=0}^{\infty} C_k g_k z^k$$

$$= G[1 - \lambda(1 - X(z))]\left\{R[V(1-\lambda)] + \frac{1}{z}\frac{[1-R[V(1-\lambda)]]}{[1-V(1-\lambda)]}\right\} \tag{8}$$

Putting the values of $A[X(z)]$ and $B[X(z)]$ from equation (7) and (8) into equation (5), we have

$$Q(z) = \pi_0 \frac{G[1-\lambda(1-X(z))]}{[G[1-\lambda(1-X(z))]-z]}\left\{\begin{array}{l} z - R[V(1-\lambda)] - \frac{[1-R[V(1-\lambda)]]}{[1-V(1-\lambda)]}\,. \\ .[V[1-\lambda(1-X(z))] - V(1-\lambda)] \end{array}\right\} \tag{9}$$

The normalizing condition for the system is

$$\sum_{k=0}^{\infty} \pi_k = 1 \tag{10}$$

By using equation (10) and L' Hospital's rule in equation (9), we have

$$\pi_0 = \frac{(1-\rho)}{R[V(1-\lambda)] - 1 + \frac{[1-R[V(1-\lambda)]]}{[1-V(1-\lambda)]}\lambda E[V]} \tag{11}$$

Putting the value of π_0 from equation (11) into (9), we have

$$Q(z) = Q_1(z).Q_2(z) \tag{12}$$

where

$$Q_1(z) = \frac{(1-\rho)\ G[1-\lambda(1-X(z))][1-X(z)]}{\lambda\ [G[1-\lambda(1-X(z))]-z]} \tag{13}$$

which is well known steady state queue length for $Geo^X/G/1$ queueing system without vacation, and

$$Q_2(z) = \frac{\lambda[z-R[V(1-\lambda)]].[1-V(1-\lambda)].[1-R[V(1-\lambda)]]}{\alpha\,[1-X(z)]} \tag{14}$$

which is steady state queue length distribution for the number of customers arriving in the queue, when server is on vacation, where,

$$\alpha = R[V(1-\lambda)] - 1 + \frac{[1-R[V(1-\lambda)]]}{[1-V(1-\lambda)]}.\lambda E[V] \tag{15}$$

Equation (12) states that the probability generating function $Q(z)$ of queue length for the system possesses Stochastic decomposition property.

3.2Waiting Time Distribution

Let us define the probability generating function $W(z)$ for the waiting time of an arrival in the system. Following (Zhang and Tian (2001)), Let A_i be the $i-th$ customer in the system

$$A_{i+1} = \begin{cases} A_i - T + S & ;(A_i - T) \geq 0 \\ \omega + S & ;(A_i - T) < 0 \end{cases} \tag{16}$$

where ω is the waiting time for the first customer of a busy period.

Define the PGF $X_k(z)$ of the arrival in the batch with mean λ_k as

$$X_k(z) = \lambda(0) + [1 - \lambda(0)]X(z) \tag{17}$$

and

$$\lambda_k = 1 - \lambda(0) \tag{18}$$

Now, from equation (12)- (18), we have

$$Q_k(z) = \frac{(1-\rho)G[1-\lambda_k(1-X_k(z))]}{[G[1-\lambda_k(1-X_k(z))-z]]}\cdot\frac{[z-R_k[V_k(1-\lambda_k)]]-[1-R_k[V_k(1-\lambda_k)]]}{\alpha} \tag{19}$$

where α is given by equation (15).

Since we know that for $Geo^X/G/1$ queue with FCFS discipline, the number of arrivals present in the system at the end of each service cycle is equal to the number of customers who join the system in the given time interval T for successive arrivals. Therefore the distribution of arrivals in the system will be equal for A_k and A_{k+1}, the customers present in the system at the each of $k-th$ service cycle and the customer arrive during the $k-th$ vacation period respectively.

From equation (16), we have, (see Zhang and Tian (2001))

$$A(z) = P_r[A \geq T]\ E[z^{A-T} \mid A \geq T]\ E[z^S] + P_r[A < T]\ E[z^{\omega} \mid Delay\ cycle] \times E[z^{\omega} \mid Service\ \ cycle] \tag{20}$$

where

$$P_r[A \geq T] = A(1-\lambda) \tag{21}$$

and

$$E[z^{A-T} \mid A \geq T] = \frac{1}{1-A(1-\lambda)} \sum_{k=0}^{\infty} z^k \sum_{m=n+1}^{\infty} C_m\ P(A=m)\ (1-\lambda)^{k-m-1}\ \lambda$$

$$= \frac{1}{1-A(1-\lambda)} \sum_{k=0}^{m-1} (1-\lambda)^{k-m-1}\ z^k\ C_k \sum_{m=n+1}^{\infty} \lambda\ C_m\ P(A=m)$$

$$= \frac{1}{1-A(1-\lambda)}\ \frac{\lambda}{[(1-\lambda)-X(z)]}\ [A(1-\lambda) - A(X(z))]$$

(22)

$$E[z^{\omega} \mid Delay\ cycle] = \sum_{k=0}^{\infty} E(z^{\omega} \mid m)\ \frac{E[S_k]}{E[V_k]}$$

$$= \frac{\sum_{k=0}^{\infty}[G_{k+1}[1-\lambda_k(1-X_k(z))]] - [G_k[1-\lambda_k(1-X_k(z))]]}{E[V_k]\,[X_k[R_k(V_k(1-\lambda_k))] - z]} \tag{23}$$

and

$$E[z^S \mid Service\ cycle] = \sum_{k=0}^{\infty} E(z^S \mid m) \tag{24}$$

Hence from equation (18) – (21), we have

$$A(z) = \frac{A(1-\lambda)[\lambda - [X(z)-(1-\lambda)]]\ \omega(X(z))G(X(z))}{[\lambda G(X(z)) - X(z) + (1-\lambda)]} \tag{25}$$

Since service time and waiting time are independent. Hence, we have

$$Q_k(z) = W_k(z).A_k(z) \tag{26}$$

From equation (19), (23) and (24), we have

$$W_k(z) = \frac{(1-\rho)[1-V_k(1-X_k(z))]\,[z-R_k(V_k(1-\lambda_k))]}{[G[1-\lambda_k(1-X_k(z))-z]]\,\lambda\,E[V_k]\,[1-R_k(V_k(1-\lambda_k))]} \tag{27}$$

which is also written as

$$W(z) = \chi_k(z)\ W_k(z) \tag{28}$$

where

$$\chi_k(z) = \frac{[1-V_k(1-X_k(z))]}{E[V_k]\,[1-X_k(z)]} \tag{29}$$

and

$$W_k(z) = \frac{(1-\rho)[z-R_k(V_k(1-\lambda_k))]\,[1-X_k(z)]}{\lambda\,[G[1-\lambda_k(1-X_k(z))-z]].[1-R_k(V_k(1-\lambda_k))]} \tag{30}$$

We consider the waiting time of a customer between a time slot in the system during each service cycle which is equal to the customers waiting time arriving during each vacation period. Hence we may write

$$\chi_k(z) = \chi(z) \tag{31}$$

From equation (27), (28),(29) and (30), we have

$$W(z) = \frac{[1-V(1-X(z))]}{E[V][1-X(z)]}\ W_{Geo^X/G/1}(z) \tag{32}$$

where $W_{Geo^X/G/1}(z)$ is well known waiting time distribution for $Geo^X/G/1$ queueing system without vacation and (32) is due to the vacations. Hence we see that the waiting time distribution is also possesses Stochastic decomposition property.

IV. EARLY ARRIVAL MODEL

Let L_k denote the number of customers arriving in the system just after the service completion of $k-th$ customer in the system. Let $\{L_k : k \geq 1\}$ be an embedded Markov chain. Then by following [Takagi (1993)], we have

$$L_{k+1} = \begin{cases} L_k + N_{n+1} - 1 & ; L_k \geq 1 \\ F_k & ; L_k \geq 0 \end{cases} \tag{33}$$

where N_{n+1} and F_k denote the number of arrivals during the $(n+1)th$ service cycle and the numbers of arrivals at least one in the system just before the service completion of $(n-1)-th$ service cycle during the customer service.

4.1 Queue Length Distribution

Define the probability distribution for F_i as

$$P_r[F_i = k] = \sum_{i=k+1}^{\infty} \binom{k-1}{k} \lambda^k \ (1-\lambda)^{i-1-k} \ g_k \ c_k \tag{34}$$

From equation (33) and (34), we have the following generating function

$$F(z) = \sum_{j=0}^{\infty} [X(z)]^j \ \sum_{i=k+1}^{\infty} \binom{k-1}{k} \lambda^k \ (1-\lambda)^{i-1-k} \ g_k \ c_k \tag{35}$$

On solving after some manipulation, we have

$$F(z) = \frac{B[1-\lambda+\lambda X(z)][1-R[V(1-\lambda)]]}{[1-\lambda+\lambda X(z)]\,[1-V(1-\lambda)]} \tag{36}$$

Using probabilistic arguments and normalizing condition from equation (10), we have

$$\pi_j = \pi_0 \, b_j + \sum_{k=1}^{j-1} \pi_k \ a_{j+1-k} \ c_k \qquad ; j > 1 \tag{37}$$

From equation (10), (36) and (37), we have

$$U(z) = \frac{(1-\rho)\,[1-X(z)]\left[F\left(1-\lambda\left(1-X(z)\right)\right)\right]\,[z-R[V(1-\lambda)]]\,[1-V(1-\lambda)]}{[1-\lambda(1-X(z))]\left[F\left(1-\lambda\left(1-X(z)\right)\right)-z\right][1-R[V(1-\lambda)]]\ \lambda E[V]} \tag{38}$$

which can be rewritten as

$$U(z) = U_1(z)\,.U_2(z) \tag{39}$$

where

$$U_1(z) = \frac{(1-\rho)\,[1-X(z)]\left[F\left(1-\lambda\left(1-X(z)\right)\right)\right]}{[1-\lambda(1-X(z))]\left[F\left(1-\lambda\left(1-X(z)\right)\right)-z\right]} \tag{40}$$

which is well known steady state queue length distribution for $Geo^X/G/1$ for the early arrival queueing system without vacation.

And

$$U_2(z) = \frac{[1-V(1-\lambda)]\ [z-R[V(1-\lambda)]]}{\lambda E[V]\,[1-R[V(1-\lambda)]]} \tag{41}$$

which is steady state queue length distribution for the number of customers arriving in the queue, when server is on vacation.

Hence, we prove that the equation (39) also follows Stochastic decomposition property for queue length distribution.

4.2Waiting Time Distribution

Let us define the probability generating function $W_k(z)$ for the waiting time of an arrival of a customer who join the queue just before the service completion of the last customer served in the $k-th$ service cycle. So, it is a case of virtual waiting time which is is equivalent to the waiting time of a customer in that service cycle.

Now from equation (18) and (19), we have, the joint PGF for the queue length as

$$D(x,z) = \frac{(1-\rho)\, z\, x \left[[1-\lambda(0)]\left[z-V\left[B\left(X(z)\right)\right]\right]+V[z\,\lambda(0)]\right]}{\left[[1-\lambda(0)]\lambda(0)\, E[V]+V[\lambda(0)]\right][1-B(X(z))]} \cdot \frac{\left[1-B\left[x\,B[X(z)]\right]\right]\left[1-V\left[B[X(z)]\right]\right]}{[B[X(z)]-z]\;[x-\lambda(0)E[V]]} \tag{42}$$

From equation (20), (21) and (42), we have

$$W_k(z) = \frac{(1-\rho)\, z\, x \left[[1-\lambda(0)]\left[z-V\left[B_k\left(X_k(z)\right)\right]\right]+V[z\,\lambda(0)]\right]}{\left[[1-\lambda(0)]\lambda(0)\, E[V]+V[\lambda(0)]\right][1-B_k(X_k(z))]} \cdot \frac{\left[1-B_k\left[x\,B_k[X_k(z)]\right]\right]\left[1-V\left[B_k[X_k(z)]\right]\right]}{[B_k[X_k(z)]-z]\;[x-\lambda(0)E[V]]} \tag{43}$$

which is rewritten as

$$W(z) = \frac{(1-\rho)(1-z)\left[1-B\left[x\,B[X(z)]\right]\right]}{\lambda\,[1-B[X(z)]]\,[B[X(z)-z]]} \cdot \frac{\lambda\, z\, x\left[[1-\lambda(0)]\left[z-V\left[B[X(z)]\right]\right]+V[z\,\lambda(0)]\right]}{(1-\lambda)\left[[1-\lambda(0)]\,\lambda(0)E[V]+V[\lambda(0)]\right][x-\lambda(0)E[V]]} \tag{44}$$

or,

$$W(z) = W_{Geo^X/G/1} \cdot \chi(z) \tag{45}$$

where first term is well known waiting time distribution for $Geo^X/G/1$ queueing system for elapsed service time and $\chi(z)$ is due to vacation. Hence it follows again Stochastic decomposition.

V. CONCLUSIONS

This paper analyses a discrete time batch arrival queueing system with multiple vacations. The Stochastic decomposition of the performance measures states that the ordinary $Geo^X/G/1$ queueing system practically independent to the vacation model due to either system failure or system goes on vacation. Here each performance measures may decompose into two terms as one for ordinary system and other due to the vacations denoted by $\chi(z)$.

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