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A Ring-Shaped Region Containing All or A Specific Number of The Zeros of A Polynomial

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ABSTRACT: According to a Cauchy's classical result all the zeros of a polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ of degree n lie in $|z|\le 1+A$, where $A=\max_{0\le j\le n-1}\left|\frac{a_j}{a_n}\right|$. In this paper we find a ring-shaped region containing all or a specific number of zeros of P(z).

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I. INTRODUCTION

A classical result giving a region containing all the zeros of a polynomial is the following known as Cauchy's Theorem [4]:

Theorem A. All the zeros of the polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ of degree n lie in the circle

$|z|\le 1+M$, where $M=\max_{0\le j\le n-1}\left|\frac{a_j}{a_n}\right|$.

The above result has been generalized and improved in various ways. Mohammad [5] used Schwarz Lemma and proved the following result:

Theorem B. All the zeros of the polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ of degree n lie in $|z|\le \frac{M}{|a_n|}$ if $|a_n|\le M$,

where $M=\max_{|z|=1}\left|a_n z^{n-1}+......+a_0\right|=\max_{|z|=1}\left|a_0 z^{n-1}+......+a_{n-1}\right|$.

Recently Gulzar [3] proved the following result :

Theorem C. All the zeros of the polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ of degree n lie in the ring-shaped region $r_1\le |z|\le r_2$, where

$$r_1=\frac{[R^4|a_1|^2(M-|a_0|)^2+4|a_1|^2R^2M^3]^{\frac{1}{2}}-|a_1|R^2(M-|a_0|)}{2M^2},$$

$$r_2=\frac{2M_1^{\,2}}{[R^4|a_{n-1}|^2(M_1-|a_n|)^2+4|a_n|R^2M^3]^{\frac{1}{2}}-|a_{n-1}|R^2(M_1-|a_n|)},$$

$$M=\max_{|z|=R}\left|a_n z^n+......+a_1 z\right|,$$
$$M_1=\max_{|z|=R}\left|a_0 z^n+......+a_{n-1} z\right|,$$

R being any positive number.

Theorem C. The number of zeros of the polynomial $P(z)=\sum_{j=0}^{n}a_j z^j$ of degree n in the ring-shaped region $r_1 \le |z| \le \frac{R}{c}, 1 < c \le R$,does not exceed

$$\frac{1}{\log c}\log(1+\frac{M}{|a_0|}),$$

where M is as in Theorem 1.

II. MAIN RESULTS

In this paper we prove the following results:

Theorem 1. Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n and let $L=\sum_{j=1}^{n}|a_j|, L'=\sum_{j=0}^{n-1}|a_j|$. Then all the zeros of P(z) lie in $r_1 \le |z| \le r_2$, where for $R \ge 1$,

$$r_1 = \frac{[R^4|a_1|^2(R^nL-|a_0|)^2+4|a_0|R^{3n+2}L^3]^{\frac{1}{2}}-|a_1|R^2(R^nL-|a_0|)}{2R^{2n}L^3|a_0|},$$

$$r_2 = \frac{2R^{2n}L'^3|a_0|}{[R^4|a_1|^2(R^nL'-|a_0|)^2+4|a_0|R^{3n+2}L'^3]^{\frac{1}{2}}-|a_1|R^2(R^nL'-|a_0|)}$$

and for $R \le 1$

$$r_1 = \frac{[|a_1|^2(RL-|a_0|)^2+4|a_0|RL^3]^{\frac{1}{2}}-|a_1|(RL-|a_0|)}{2RL^3|a_0|},$$

$$r_2 = \frac{2RL'^3|a_0|}{[|a_1|^2(RL'-|a_0|)^2+4|a_0|RL'^3]^{\frac{1}{2}}-|a_1|(RL'-|a_0|)},$$

R being any positive number.

Theorem2. Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n and let $L=\sum_{j=1}^{n}|a_j|$. Then the number of zeros of P(z) in $r_1 \le |z| \le \frac{R}{c}, 1 < c$ does not exceed $\frac{1}{\log c}\log\frac{R^nL+|a_0|}{|a_0|}=\frac{1}{\log c}\log(1+\frac{R^nL}{|a_0|})$ for $R \ge 1$ and $\frac{1}{\log c}\log\frac{RL+|a_0|}{|a_0|}=\frac{1}{\log c}\log(1+\frac{RL}{|a_0|})$ for $R \le 1$.

For different values of the parameters in Theorems 1 and 2, we get many interesting results. For example for R=1, we get the following results:

Corollary 1. Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n and let $L=\sum_{j=1}^{n}|a_j|, L'=\sum_{j=0}^{n-1}|a_j|$. Then all the zeros of P(z) lie in $r_1 \le |z| \le r_2$, where,

$$r_1 = \frac{[|a_1|^2(L-|a_0|)^2 + 4|a_0|L^3]^{\frac{1}{2}} - |a_1|(L-|a_0|)}{2L^3|a_0|},$$

$$r_2 = \frac{2L'^3|a_0|}{[|a_1|^2(L'-|a_0|)^2 + 4|a_0|L'^3]^{\frac{1}{2}} - |a_1|(L'-|a_0|)}.$$

Corollary 2. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n and let $L=\sum_{j=1}^{n}|a_j|$. Then the number of zeros of P(z) in $r_1 \le |z| \le \frac{R}{c}, 1<c$ does not exceed $\frac{1}{\log c}\log\frac{L+|a_0|}{|a_0|} = \frac{1}{\log c}\log(1+\frac{L}{|a_0|})$.

III. LEMMAS

For the proofs of the above theorems we make use of the following results:

Lemma1. Let f(z) be analytic in $|z|\le 1$,f(0)=a, where $|a|<1, f'(0)=b$ and $|f(z)|\le 1$for $|z|=1$. Then, for $|z|\le 1$,

$$|f(z)| \le \frac{(1-|a|)|z|^2 + |b||z| + |a|(1-|a|)}{|a|(1-|a|)|z|^2 + |b||z| + (1-|a|)}.$$

The example

$$f(z) = \frac{a + \frac{b}{1+a}z - z^2}{1 - \frac{b}{1+a}z - az^2}$$

shows that the estimate is sharp.

Lemma 1 is due to Govil, Rahman and Schmeisser [2].

Lemma 2. Let f(z) be analytic for $|z|\le R$,f(0)=0, $f'(0)=b$ and $|f(z)|\le M$ for $|z|=R$.

Then, for $|z|\le R$,

$$|f(z)| \le \frac{M|z|}{R^2}.\frac{M|z|+R^2|b|}{M+|b||z|}.$$

Lemma 2 is a simple deduction from Lemma 1 .

Lemma 3. If f(z) is analytic for $|z|\le R$, $f(0)\ne 0$ and $|f(z)|\le M$ for $|z|=R$, then the number of zeros of f(z)in $|z|\le \frac{R}{c}, 1<c<R$ is less than or equal to $\frac{1}{\log c}\log\frac{M}{|f(0)|}$.

Lemma 3 is a simple deduction from Jensen's Theorem (see [1]).

IV. PROOFS OF THEOREMS

Proof of Theorem 1. Let G(z)= $a_n z^n + a_{n-1}z^{n-1} + \ldots\ldots + a_1 z$.

Then G(z) is analytic for $|z|\le R$,G(0)=0, $G'(0)=a_1$and for $|z|=R$

$$|G(z)| = |a_n z^n + a_{n-1}z^{n-1} + \ldots\ldots + a_1 z|$$

$$\le |a_n||z|^n + |a_{n-1}||z|^{n-1} + \ldots\ldots + |a_1||z|$$
$$= |a_n|R^n + |a_{n-1}|R^{n-1} + \ldots\ldots + |a_1|R$$
$$\le R^n[|a_n| + |a_{n-1}| + \ldots\ldots + |a_1|]$$
$$= R^n L$$

for $R \ge 1$
and for $R \le 1$,

$$|G(z)| \le RL.$$

Hence, by Lemma 2, for $|z| \le R$,

$$|G(z)| \le \frac{R^n L|z|}{R^2} . \frac{R^n L|z| + |a_1|R^2}{R^n L + |a_1||z|},$$

for $R \ge 1$
and for $R \le 1$,

$$|G(z)| \le \frac{RL|z|}{R^2} . \frac{RL|z| + |a_1|R^2}{RL + |a_1||z|}.$$

Therefore, for $|z| \le R$, $R \ge 1$,

$$|P(z)| = |G(z) + a_0|$$
$$\ge |a_0| - |G(z)|$$
$$\ge |a_0| - \frac{R^n L|z|}{R^2} . \frac{R^n L|z| + |a_1|R^2}{R^n L + |a_1||z|}$$
$$> 0$$

if

$$|a_0|R^2(R^n L + |a_1||z|) - R^n L|z|(R^n L|z| + |a_1|R^2) > 0$$

i.e. if

$$R^{2n}L^2|z|^2 + |a_1|R^2(R^n L - |a_0|)|z| - |a_0|R^{n+2}L < 0$$

which is true if

$$|z| < \frac{[R^4|a_1|^2(R^n L - |a_0|)^2 + 4|a_0|R^{3n+2}L^3]^{\frac{1}{2}} - |a_1|R^2(R^n L - |a_0|)}{2R^{2n}L^3|a_0|}.$$

Similarly, for $|z| \le R, R \le 1, |P(z)| > 0$ if

$$|a_0|R^2(RL + |a_1||z|) - RL|z|(RL|z| + |a_1|R^2) > 0$$

i.e. if

$$L^2|z|^2 + |a_1|(RL - |a_0|)|z| - |a_0|RL < 0$$

which is true if

$$|z| < \frac{[|a_1|^2(RL - |a_0|)^2 + 4|a_0|RL^3]^{\frac{1}{2}} - |a_1|(RL - |a_0|)}{2RL^3|a_0|}.$$

This shows that P(z) does not vanish in

$$|z| < \frac{[R^4|a_1|^2(R^nL-|a_0|)^2+4|a_0|R^{3n+2}L^3]^{\frac{1}{2}}-|a_1|R^2(R^nL-|a_0|)}{2R^{2n}L^3|a_0|}$$

for $|z| \le R$, $R \ge 1$

and P(z) does not vanish in

$$|z| < \frac{[|a_1|^2(RL-|a_0|)^2+4|a_0|RL^3]^{\frac{1}{2}}-|a_1|(RL-|a_0|)}{2RL^3|a_0|}$$

for $|z| \le R$, $R \le 1$.

In other words, all the zeros of P(z) lie in

$$|z| \ge \frac{[R^4|a_1|^2(R^nL-|a_0|)^2+4|a_0|R^{3n+2}L^3]^{\frac{1}{2}}-|a_1|R^2(R^nL-|a_0|)}{2R^{2n}L^3|a_0|}$$

i.e. $|z| \ge r_1$ for $|z| \le R$, $R \ge 1$

and in

$$|z| \ge \frac{[|a_1|^2(RL-|a_0|)^2+4|a_0|RL^3]^{\frac{1}{2}}-|a_1|(RL-|a_0|)}{2RL^3|a_0|}$$

i.e. $|z| \ge r_1$ for $|z| \le R$, $R \le 1$.

On the other hand, let

$$Q(z) = z^nP(\frac{1}{z}) = a_0z^n + a_1z^{n-1} + \ldots\ldots + a_{n-1}z + a_n$$
$$= H(z) + a_n,$$

where

$$H(z) = a_0z^n + a_1z^{n-1} + \ldots\ldots + a_{n-1}z\,.$$

Then H(z) is analytic and $|H(z)| \le R^nL'$ for $|z| = R$, H(0)=0, $H'(0) = a_{n-1}$. Hence, by Lemma 2, for $|z| \le R$,

$$|H(z)| \le \frac{R^nL'|z|}{R^2}.\frac{R^nL'|z|+|a_1|R^2}{R^nL'+|a_1||z|},$$

for $R \ge 1$

and for $R \le 1$,

$$|H(z)| \le \frac{RL'|z|}{R^2}.\frac{RL'|z|+|a_1|R^2}{RL'+|a_1||z|}..$$

Therefore, for $|z| \le R$, $R \ge 1$,

$$|Q(z)| = |H(z)+a_0|$$
$$\ge |a_0| - |H(z)|$$
$$\ge |a_0| - \frac{R^nL'|z|}{R^2}.\frac{R^nL'|z|+|a_1|R^2}{R^nL\ \ +|a_1||z|}$$
$$> 0$$

if

$$|a_0|R^2(R^nL' + |a_1||z|) - R^nL'|z|(R^nL'|z| + |a_1|R^2) > 0$$

i.e. if

$$R^{2n}L'^2|z|^2 + |a_1|R^2(R^nL' - |a_0|)|z| - |a_0|R^{n+2}L' < 0$$

which is true if

$$|z| < \frac{[R^4|a_1|^2(R^nL' - |a_0|)^2 + 4|a_0|R^{3n+2}L'^3]^{\frac{1}{2}} - |a_1|R^2(R^nL' - |a_0|)}{2R^{2n}L'^3|a_0|}.$$

Similarly, for $|z| \le R$, $R \le 1$, $|Q(z)| > 0$ if

$$|a_0|R^2(RL' + |a_1||z|) - RL'|z|(RL'|z| + |a_1|R^2) > 0$$

i.e. if

$$L'^2|z|^2 + |a_1|(RL' - |a_0|)|z| - |a_0|RL' < 0$$

which is true if

$$|z| < \frac{[|a_1|^2(RL' - |a_0|)^2 + 4|a_0|RL'^3]^{\frac{1}{2}} - |a_1|(RL' - |a_0|)}{2RL'^3|a_0|}.$$

This shows that Q(z) does not vanish in

$$|z| < \frac{[R^4|a_1|^2(R^nL' - |a_0|)^2 + 4|a_0|R^{3n+2}L'^3]^{\frac{1}{2}} - |a_1|R^2(R^nL' - |a_0|)}{2R^{2n}L'^3|a_0|}$$

for $|z| \le R$, $R \ge 1$

and Q(z) does not vanish in

$$|z| < \frac{[|a_1|^2(RL' - |a_0|)^2 + 4|a_0|RL'^3]^{\frac{1}{2}} - |a_1|(RL' - |a_0|)}{2RL'^3|a_0|}$$

for $|z| \le R$, $R \le 1$.

In other words, all the zeros of $Q(z) = z^nP(\frac{1}{z})$ lie in

$$|z| \ge \frac{[R^4|a_1|^2(R^nL' - |a_0|)^2 + 4|a_0|R^{3n+2}L'^3]^{\frac{1}{2}} - |a_1|R^2(R^nL' - |a_0|)}{2R^{2n}L'^3|a_0|}$$

for $|z| \le R$, $R \ge 1$

and in

$$|z| \ge \frac{[|a_1|^2(RL - |a_0|)^2 + 4|a_0|RL^3]^{\frac{1}{2}} - |a_1|(RL - |a_0|)}{2RL^3|a_0|}$$

for $|z| \le R$, $R \le 1$.

Hence all the zeros of $P(\frac{1}{z})$ lie in

$$|z| \ge \frac{[R^4|a_1|^2(R^nL' - |a_0|)^2 + 4|a_0|R^{3n+2}L'^3]^{\frac{1}{2}} - |a_1|R^2(R^nL' - |a_0|)}{2R^{2n}L'^3|a_0|}$$

for $|z| \le R,\ R \ge 1$

and in

$$|z| \ge \frac{[|a_1|^2 (RL' - |a_0|)^2 + 4|a_0| RL'^3]^{\frac{1}{2}} - |a_1|(RL' - |a_0|)}{2RL'^3 |a_0|}$$

for $|z| \le R,\ R \le 1$.

Therefore , all the zeros of P(z) lie in

$$|z| \le \frac{2R^{2n} L'^3 |a_0|}{[R^4 |a_1|^2 (R^n L' - |a_0|)^2 + 4|a_0| R^{3n+2} L'^3]^{\frac{1}{2}} - |a_1| R^2 (R^n L' - |a_0|)}$$

i.e. $|z| \le r_2$ for $|z| \le R,\ R \ge 1$ and in

$$|z| \le \frac{2RL'^3 |a_0|}{[|a_1|^2 (RL' - |a_0|)^2 + 4|a_0| RL'^3]^{\frac{1}{2}} - |a_1|(RL' - |a_0|)}$$

i.e. $|z| \le r_2$ for $|z| \le R,\ R \le 1$.

Thus we have proved that all the zeros of P(z) lie in $r_1 \le |z| \le r_2$. That completes the proof of Theorem 1.

Proof of Theorem 2. To prove Theorem 2, we need to show only that the number of zeros of P(z) in $|z| \le \frac{R}{c}, 1 < c$ does not exceed $\frac{1}{\log c} \log(1 + \frac{R^n L}{|a_0|})$ for $R \ge 1$ and $\frac{1}{\log c} \log(1 + \frac{RL}{|a_0|})$ for $R \le 1$.

Since P(z) is analytic for $|z| \le R$, P(0)= a_0 and for $|z| \le R$,

$$\begin{aligned} |P(z)| &= |a_n z^n + a_{n-1} z^{n-1} + \dots\dots + a_1 z + a_0| \\ &\le |a_n||z|^n + |a_{n-1}||z|^{n-1} + \dots\dots + |a_1||z| + |a_0| \\ &\le |a_n| R^n + |a_{n-1}| R^{n-1} + \dots\dots + |a_1| R + |a_0| \\ &\le R^n L + |a_0| \end{aligned}$$

for $R \ge 1$

and

$$|P(z)| \le RL + |a_0|$$

for $R \le 1$,

it follows , by Lemma 3, that the number of zeros of P(z) in $r_1 \le |z| \le \frac{R}{c}, 1 < c$ does not exceed

$$\frac{1}{\log c} \log \frac{R^n L + |a_0|}{|a_0|} = \frac{1}{\log c} \log(1 + \frac{R^n L}{|a_0|}) \text{ for } R \ge 1$$

and $\frac{1}{\log c} \log \frac{RL + |a_0|}{|a_0|} = \frac{1}{\log c} \log(1 + \frac{RL}{|a_0|})$ for $R \le 1$ and Theorem 2 follows.

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