On the Zeros of A Polynomial Inside the Unit Disc

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ABSTRACT: In this paper we find the number of zeros of a polynomial inside the unit disc under certain conditions on the coefficients of the polynomial.

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I. INTRODUCTION

In the context of the Enestrom-Kakeya Theorem [4] which states that all the zeros of a polynomial

\[ P(z) = \sum_{j=0}^{n} a_j z^j \] with \( a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0 \) lie in \( |z| \leq 1 \), Q. G. Mohammad [5] proved the following result giving a bound for the number of zeros of \( P(z) \) in \( |z| \leq \frac{1}{2} \):

**Theorem A:** Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) such that

\[ a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0, \]

Then the number of zeros of \( P(z) \) in \( |z| \leq \frac{1}{2} \) does not exceed

\[ 1 + \frac{1}{\log 2} \log \frac{a_n}{a_0}. \]

Various bounds for the number of zeros of a polynomial with certain conditions on the coefficients were afterwards given by researchers in the field (e.g. see [1],[2],[3]).

II. MAIN RESULTS

In this paper we find a bound for the number of zeros of a polynomial in a closed disc of radius less than \( 1 \) and prove

**Theorem 1:** Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) with \( \text{Re}(a_j) = \alpha_j, \text{Im}(a_j) = \beta_j, j = 0,1,2, \ldots, n \) such that for some \( \lambda, 0 \leq \lambda \leq n - 1 \) and for some \( k \geq 1, 0 < \tau \leq 1 \),

\[ k\alpha_n \geq \alpha_{n-1} \geq \ldots \geq \alpha_{k+1} \geq \tau \alpha_k \]

and

\[ L = |\alpha_k - \alpha_{k+1}| + |\alpha_{k-1} - \alpha_{k+2}| + \ldots + |\alpha_1 - \alpha_0| + |\alpha_0|. \]

Then the number of zeros of \( P(z) \) in \( |z| \leq \delta, 0 < \delta < 1 \) does not exceed

\[ \frac{1}{\log \frac{1}{\delta}} \left( \frac{1}{|a_n|} + (k-1)|\alpha_k| + k\alpha_n - \tau \alpha_k + L + (1-\tau)|\alpha_k| + 2 \sum_{j=0}^{n} |\beta_j| \right). \]

Taking \( a_j \) real i.e. \( \beta_j = 0, \forall j = 0,1,2, \ldots, n \), Theorem 1 reduces to the following result:

**Corollary 1:** Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) such that for some \( \lambda, 0 \leq \lambda \leq n - 1 \) and for some \( k \geq 1, 0 < \tau \leq 1 \),

\[ k\alpha_n \geq \alpha_{n-1} \geq \ldots \geq \alpha_{k+1} \geq \tau \alpha_k \]
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and

\[ L = |a_\lambda - a_{\lambda-1}| + |a_{\lambda-1} - a_{\lambda-2}| + \ldots + |a_1 - a_0| + |a_0| \]

Then the number of zeros of \( P(z) \) in \( |z| \leq \delta, 0 < \delta < 1 \) does not exceed

\[ \frac{1}{\log \frac{1}{\delta}} \log \frac{k(a_n + a_n - \lambda a_\lambda + L + (1 - \tau)|a_\lambda|)}{|a_0|} \].

Taking \( \tau = 1 \) in Cor. 1, we get the following result:

**Corollary 2:** Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) such that for some \( \lambda, 0 \leq \lambda \leq n - 1 \) and for some \( k \geq 1 \),

\[ k a_n \geq a_{n-1} \geq \ldots \geq a_{\lambda+1} \geq a_\lambda \]

and

\[ L = |a_\lambda - a_{\lambda-1}| + |a_{\lambda-1} - a_{\lambda-2}| + \ldots + |a_1 - a_0| + |a_0| \]

Then the number of zeros of \( P(z) \) in \( |z| \leq \delta, 0 < \delta < 1 \) does not exceed

\[ \frac{1}{\log \frac{1}{\delta}} \log \frac{k(a_n + a_n - \lambda a_\lambda + L)}{|a_0|} \].

Taking \( k = 1 \) in Cor. 1, we get the following result:

**Corollary 3:** Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) such that for some \( \lambda, 0 \leq \lambda \leq n - 1 \) and for some \( o < \tau \leq 1 \),

\[ a_n \geq a_{n-1} \geq \ldots \geq a_{\lambda+1} \geq \tau a_\lambda \]

and

\[ L = |a_\lambda - a_{\lambda-1}| + |a_{\lambda-1} - a_{\lambda-2}| + \ldots + |a_1 - a_0| + |a_0| \]

Then the number of zeros of \( P(z) \) in \( |z| \leq \delta, 0 < \delta < 1 \) does not exceed

\[ \frac{1}{\log \frac{1}{\delta}} \log \frac{|a_n + a_n - \tau a_\lambda + L + (1 - \tau)|a_\lambda|)}{|a_0|} \].

Taking \( \tau = 1 \) in Theorem 1, we get the following result:

**Corollary 4:** Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) with \( \text{Re}(a_j) = \alpha_j, \text{Im}(a_j) = \beta_j \),

\( j = 0, 1, 2 \ldots \ldots n \) such that for some \( \lambda, 0 \leq \lambda \leq n - 1 \) and for some \( k \geq 1 \),

\[ k a_n \geq a_{n-1} \geq \ldots \geq a_{\lambda+1} \geq \alpha_\lambda \]

and

\[ L = |\alpha_\lambda - a_{\lambda-1}| + |\alpha_{\lambda-1} - a_{\lambda-2}| + \ldots + |\alpha_1 - a_0| + |a_0| \]

Then the number of zeros of \( P(z) \) in \( |z| \leq \delta, 0 < \delta < 1 \) does not exceed

\[ \frac{1}{\log \frac{1}{\delta}} \log \frac{|a_n + (k-1)| \alpha_n| + k a_n - \alpha_\lambda + L + 2 \sum_{j=0}^{n} |\beta_j|}{|a_0|} \].

Taking \( k = 1 \) in Theorem 1, we get the following result:
Corollary 5: Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) with \( \text{Re}(a_j) = \alpha_j, \text{Im}(a_j) = \beta_j \), \( j = 0, 1, 2, \ldots, n \) such that for some \( \lambda, 0 \leq \lambda \leq n - 1 \) and for some \( 0 < \tau \leq 1 \),
\[
\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_{\lambda+1} \geq \tau \alpha_{\lambda},
\]
and
\[
L = |\alpha_{\lambda} - \alpha_{\lambda-1}| + |\alpha_{\lambda-1} - \alpha_{\lambda-2}| + \cdots + |\alpha_1 - \alpha_0| + |\alpha_0|.
\]
Then the number \( f \) zeros of \( P(z) \) in \( |z| \leq \delta, 0 < \delta < 1 \) does not exceed
\[
\frac{1}{\log \frac{1}{\delta}} \log \frac{|\alpha_0|}{|a_n| + \alpha_n - \alpha_\lambda + L + (1 - \tau)|\alpha_\lambda| + 2 \sum_{j=0}^{n} |\beta_j|}.
\]
Similarly for other different values of the parameters, we get many other interesting results.

III. LEMMA

For the proof of Theorem 1, we need the following result:

Lemma: Let \( f(z) \) be analytic for \( |z| \leq 1, f(0) \neq 0 \) and \( |f(z)| \leq M \) for \( |z| \leq 1 \). Then the number of zeros of \( f(z) \) in \( |z| \leq \delta, 0 < \delta < 1 \) does not exceed
\[
\frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|f(0)|}
\]
(for reference see [6]).

IV. PROOF OF THEOREM 1

Consider the polynomial
\[
F(z) = (1 - z)P(z)
\]
\[
= (1 - z)(a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0)
\]
\[
= a_n z^n + (a_n - a_{n-1}) z^{n-1} + \cdots + (a_{\lambda+1} - a_{\lambda}) z^{\lambda+1} + (a_{\lambda} - a_{\lambda-1}) z^{\lambda}
\]
\[
+ \cdots + (a_1 - a_0) z + a_0
\]
\[
= a_n z^n - (k - 1) a_{n-1} z^{n-1} + (k a_{n-1} - a_{n-2}) z^{n-2} + \cdots + (a_{\lambda+1} - \tau a_{\lambda}) z^{\lambda+1}
\]
\[
+ (\tau - 1) a_{\lambda} z^{\lambda} + (a_{\lambda} - a_{\lambda-1}) z^{\lambda-1} + \cdots + (a_1 - a_0) z + a_0 + i(\beta_n - \beta_{n-1}) z^n
\]
\[
+ \cdots + (\beta_1 - \beta_0) z + \beta_0 \}
\]
For \( |z| \leq 1 \), we have, by using the hypothesis
\[
|F(z)| \leq |a_n| + (k - 1)|a_n| + k|a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_{\lambda+1} - \tau a_{\lambda}| + (1 - \tau)|\alpha_{\lambda}|
\]
\[
+ |\alpha_{\lambda} - \alpha_{\lambda-1}| + \cdots + |\alpha_1 - a_0| + |\alpha_0| + |\beta_n - \beta_{n-1}| + |\beta_{n-1} - \beta_{n-2}| + \cdots
\]
\[
+ |\beta_1 - \beta_0| + |\beta_0|
\]
\[
\leq |a_n| + (k - 1)|a_n| + k|a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_{\lambda+1} - \tau a_{\lambda}| + (1 - \tau)|\alpha_{\lambda}|
\]
\[
+ |\alpha_{\lambda} - \alpha_{\lambda-1}| + \cdots + |\alpha_1 - a_0| + |\alpha_0| + |\beta_n - \beta_{n-1}| + |\beta_{n-1} - \beta_{n-2}| + \cdots
\]
\[
+ |\beta_1 - \beta_0| + |\beta_0|
\]
\[
\leq |a_n| + (k - 1)|a_n| + k|a_n| + |\alpha_{\lambda} + L - \tau (|\alpha_{\lambda}| + |\alpha_{\lambda}|) + 2 \sum_{j=0}^{n} |\beta_j|\]

Since \( F(z) \) is analytic for \( |z| \leq 1 \), \( F(0) = a_0 \neq 0 \), it follows by the Lemma that the number of zeros
of $F(z)$ in $|z| \leq \delta, 0 < \delta < 1$ does not exceed
\[
\frac{1}{\log \frac{1}{\delta}} \log \frac{1}{\delta} \left| a_0 \right| + (k - 1)\left| a_n \right| + k\left| a_n \right| + L - \tau \left( \left| a_j \right| + \left| a_j \right| \right) + 2 \sum_{j=0}^{n} \left| \beta_j \right|
\]

Since the zeros of $P(z)$ are also the zeros of $F(z)$, it follows that the number of zeros of $P(z)$ in $|z| \leq \delta, 0 < \delta < 1$ does not exceed
\[
\frac{1}{\log \frac{1}{\delta}} \log \frac{1}{\delta} \left| a_0 \right| + (k - 1)\left| a_n \right| + k\left| a_n \right| + L - \tau \left( \left| a_j \right| + \left| a_j \right| \right) + 2 \sum_{j=0}^{n} \left| \beta_j \right|
\]

That completes the proof of Theorem 1.

REFERENCES