Five-Dimensional Finsler Spaces with T-Tensor of Some Special forms

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Abstract: The T−tensor played an important role in the Finsler geometry. In this paper, we discuss a five-dimensional Finsler space whose T−tensor is of special forms.

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I. INTRODUCTION

Let $M^5$ be a five-dimensional smooth manifold and $F^5 = (M^5, L)$ be a five-dimensional Finsler space equipped with a metric function $L(x, y)$ on $M^5$. The normalized supporting element, the metric tensor, the angular metric tensor and Cartan tensor are defined by

$$l_i = \hat{\partial}_i L, \quad g_{ij} = \frac{1}{2} \hat{\partial}_i \hat{\partial}_j L^2, \quad h_{ij} = L \hat{\partial}_i \hat{\partial}_j L \quad \text{and} \quad C_{ijk} = \frac{1}{2} \hat{\partial}_k g_{ij}$$

respectively.

The torsion vector $C^i$ is defined by $C^i = C_{jk} g^{jk}$.

Throughout the paper, we use the symbols $\hat{\partial}_i$ and $\hat{\partial}_j$ for $\partial / \partial y^i$ and $\partial / \partial x^j$ respectively. The Cartan connection in the Finsler space is given as $\Gamma^i = (F^i_{jk}, G^i_j, C^i_{jk})$. The $h$− and $v$−covariant derivatives of a covariant vector $X_i(x, y)$ with respect to the Cartan connection are given by

$$X^i_{hj} = \partial_j X_i - (\hat{\partial}_h X_i) G^i_j - F^i_{ij} X_r, \quad (1.1)$$

and

$$X^i_{v|j} = \partial_j X_i - C^i_{jk} X^k, \quad (1.2)$$

In 1972, H. Kawaguchi [1] and M. Matsumoto [2] independently found an important tensor

$$T_{hijk} = L C_{hij}^{\,\,k} + C_{hij}^{\,\,l} + C_{hik}^{\,\,l} + C_{hjl}^{\,\,k} + C_{ijk}^{\,\,h}. \quad (1.3)$$

This is called the $T$−tensor. It is completely symmetric in its indices. The vanishing of $T$−tensor is called $T$-condition.

U. P. Singh et al. [3, 4] studied three-dimensional Finsler spaces with $T$−tensor of the following forms:

(A) $T_{hijk} = \rho (h_{ij} h_{jk} + h_{hj} h_{ik} + h_{ki} h_{hj})$,

(B) $T_{hijk} = h_{ij} P_{jk} + h_{ij} P_{ik} + h_{ij} P_{hj} + h_{ij} P_{ik} + h_{ij} P_{hj} + h_{ij} P_{hj}$,

(C) $T_{hijk} = \rho C_{gh} C_{ij} C_{k} + a_n C_{ij} C_{k} + a_n C_{ij} C_{k} + a_n C_{ij} C_{k} + a_n C_{ij} C_{k} + a_n C_{ij} C_{k} + a_n C_{ij} C_{k}$,

where $P_{ij}$ are the components of a tensor field, $a_n$ are the components of a covariant vector field and $\rho$ is a scalar. Present authors studied the theory of five-dimensional Finsler space. In this paper [9-11], we discuss five-dimensional Finsler spaces with $T$−tensor of such forms.

II. FIVE-DIMENSIONAL FINSLER SPACE

The Miron frame for a five-dimensional Finsler space is constructed by the unit vectors $(\epsilon^i_1, \epsilon^i_2, \epsilon^i_3, \epsilon^i_4, \epsilon^i_5)$. The first vector $\epsilon^i_1$ is the normalized supporting element $l^i$ and the second $\epsilon^i_2$, is the
normalized torsion vector $m^i = C^i / C$, the third $e^i_{3j} = n^i$, the fourth $e^i_{4j} = p^i$ and the fifth $e^i_{5j} = q^i$ are constructed by $g_{ij} e^i_{a\beta} e^j_{\gamma\beta} = \delta_{a\beta}$. We suppose that the length $C$ of the vector $C^i$ does not vanish, i.e., the space is non-Riemannian. With respect to this frame, the scalar components of an arbitrary tensor $T^i_j$ are defined by

$$ T^i_{a\beta} = T^i_j e^i_{a\beta} e^j_{\rho\beta}, $$

(2.1)

from which, we get

$$ T^i_j = T^i_{a\beta} e^i_{a\beta} e^j_{\rho\beta}, $$

(2.2)

where summation convention is also applied to Greek indices. The scalar components of the metric tensor $g_{ij}$ are $\delta_{a\beta}$.

Let $H_{a\beta j\gamma}$ and $V_{a\beta j\gamma}|L$ be scalar components of the $h$– and $v$–covariant derivatives $e^i_{a\beta j}$ and $e^i_{a\beta j}|_L$ respectively of the vectors $e^i_{a\beta}$, then

$$ e^i_{a\beta j} = H_{a\beta j\gamma} e^\gamma_{\gamma\beta}, $$

(2.3)

and

$$ Le^i_{a\beta j} = V_{a\beta j\gamma} e^\gamma_{\gamma\beta}. $$

(2.4)

$H_{a\beta j\gamma}$ and $V_{a\beta j\gamma}$ are called $h$– and $v$–connection scalars respectively and are positively homogeneous of degree zero in $y$. Orthogonality of the Miron frame yields [5] $H_{a\beta j\gamma} = -H_{\beta j a\gamma}$ and $V_{a\beta j\gamma} = -V_{\beta j a\gamma}$. Also, we have $H_{1j\beta\gamma} = 0$ and $V_{1j\beta\gamma} = \delta_{j\beta} - \delta_{\beta j} \delta_{\gamma\gamma}$. Now, we define Finsler vector fields:

$$ h = H_{2345} e^5_{\beta\gamma}, \quad j = H_{2456} e^6_{\beta\gamma}, \quad k = H_{2567} e^7_{\beta\gamma}, $$

$$ h' = H_{3456} e^6_{\beta\gamma}, \quad j' = H_{3567} e^7_{\beta\gamma}, \quad k' = H_{4567} e^7_{\beta\gamma}, $$

and

$$ u = V_{2345} e^5_{\beta\gamma}, \quad v = V_{2456} e^6_{\beta\gamma}, \quad w = V_{2567} e^7_{\beta\gamma}, $$

$$ u' = V_{3456} e^6_{\beta\gamma}, \quad v' = V_{3567} e^7_{\beta\gamma}, \quad w' = V_{4567} e^7_{\beta\gamma}. $$

Definition. The Finsler vector fields $(h, j, k, h', j', k')$ are called $h$–connection vectors and the vector fields $(u, v, w, u', v', w')$ are called $v$–connection vectors. The scalars $H_{2345}, H_{2456}, H_{3456}, H_{3567}, H_{4567}$ and $V_{2345}, V_{2456}, V_{2567}, V_{3456}, V_{3567}, V_{4567}$ are considered as the scalar components $h_{\beta \gamma}, j_{\beta \gamma}, k_{\beta \gamma}, h'_{\beta \gamma}, j'_{\beta \gamma}, k'_{\beta \gamma}$ and $u_{\beta \gamma}, v_{\beta \gamma}, w_{\beta \gamma}, u'_{\beta \gamma}, v'_{\beta \gamma}, w'_{\beta \gamma}$ of the $h$– and $v$–connection vectors respectively with respect to the orthonormal frame.

From (2.4), we get

(a) $Le^i_{a\beta j}|_L = Lm^i |_j = m^i m^j + n^i n^j + p^i p^j + q^i q^j = h^i_j,$

(b) $Le^i_{a\beta j} |_L = Ln^i |_j = -l^i m^j + n^i u^j + p^i v^j + q^i w^j,$

(c) $Le^i_{a\beta j} |_L = Ln^i |_j = -l^i n^j - m^i u^j + p^i v^j + q^i w^j,$

(d) $Le^i_{a\beta j} |_L = Lp^i |_j = -l^i p^j - m^i v^j - n^i u^j + q^i w^j,$

(e) $Le^i_{a\beta j} |_L = Lq^i |_j = -l^i q^j - m^i w^j - n^i v^j - p^i w^j.$

(2.5)

Since $m^i, n^i, p^i, q^i$ are homogeneous function of degree zero in $y^i$, we have

$$ Ln^i |_j = Lm^i |_j = Lp^i |_j = Lq^i |_j = 0.$$

This imply $u_i = v_i = w_i = u'_i = v'_i = w'_i = 0$. Consequently, we have

Proposition 2.1. The first scalar components $u_i, v_i, w_i, u'_i, v'_i, w'_i$ of $v$–connection vectors $u_i, v_i, w_i, u'_i, v'_i, w'_i$ vanish identically.
Let $C_{\alpha\beta\gamma}$ be the scalar components of $LC_{ijk}$ with respect to the Miron frame, i.e.,

$$LC_{ijk} = C_{\alpha\beta\gamma} e^\alpha_{\mu} e^\beta_{\nu} e^\gamma_{\rho}.$$  

(2.6)

The main scalars of a five-dimensional Finsler space are given by [9-10]

$$C_{222} = H, \quad C_{233} = I, \quad C_{244} = K, \quad C_{255} = M, \quad C_{333} = J,$$

$$C_{344} = J', \quad C_{444} = H', \quad C_{334} = I', \quad C_{234} = K', \quad C_{355} = J'',$$

$$C_{455} = M', \quad C_{555} = H'',$$

$$C_{245} = N', \quad C_{345} = M''.$$  

we have

$$C_{322} = -(J + J' + J''), \quad C_{224} = -(H' + I' + M'), \quad C_{225} = -(H'' + I' + M'')$$

and

$$H + I + K + M = LC.$$  

(2.7)

The scalar components $T_{\alpha\beta\gamma}$ of $LT_{ijk}$ are written in the form [5]

$$T_{\alpha\beta\gamma} = L(\hat{\partial}_{\alpha} T_{ijk} e^k_{\gamma}) + T_{\rho\mu\nu} V_{\mu\nu} + T_{\alpha\beta\gamma} V_{\alpha\beta\gamma}.$$  

(2.8)

The explicit form of $C_{\alpha\beta\gamma}$ is obtained as follows:

$$C_{222,\delta} = H_{,\delta} + 3(J + J' + J'')u_{\delta} + 3(H' + I' + M')v_{\delta} + 3(H'' + I' + K'')w_{\delta},$$

$$C_{223,\delta} = -(J + J' + J'')_{,\delta} + (H - 2I)u_{\delta} - 2K'v_{\delta} - 2Nw_{\delta} + (H' + I' + M')u_{\delta}',$$

$$+(H'' + I' + M'')v_{\delta}',$$

$$C_{224,\delta} = -(H' + I' + M')_{,\delta} - 2K'u_{\delta} + (H - 2K)v_{\delta} - 2N'w_{\delta} - (J + J' + J'')u_{\delta}',$$

$$+(H'' + I' + K'')v_{\delta}',$$

$$C_{225,\delta} = -(H'' + I' + K'')_{,\delta} - 2Nu_{\delta} - 2N'v_{\delta} + (H - 2M)w_{\delta} - (J + J' + J'')v_{\delta}',$$

$$-(H' + I' + M')w_{\delta}',$$

$$C_{233,\delta} = I_{,\delta} - (3J + 2J' + 2J'')u_{\delta} - I'v_{\delta} - I''w_{\delta} - 2Nv_{\delta}' - 2K'u_{\delta}',$$

$$C_{234,\delta} = K_{,\delta} - (2I + H' + M')u_{\delta} - (2J' + J + J'')v_{\delta} - M''w_{\delta} - (K - I)u_{\delta}',$$

$$- N''v_{\delta}' - Nw_{\delta}' + N_{,\delta},$$

$$C_{235,\delta} = M_{,\delta} - (2I' + H'' + K'')u_{\delta} - M''v_{\delta} - (J + J' + 2J')w_{\delta} - N'v_{\delta}' - Nw_{\delta}',$$

$$- M''u_{\delta} - (M - I)v_{\delta}' + K'w_{\delta}',$$

$$C_{244,\delta} = K_{,\delta} - J'u_{\delta} - (3H' + 2I + 2M')v_{\delta} + 2Ku_{\delta}' - K''w_{\delta} - 2N'w_{\delta}',$$

$$C_{245,\delta} = N_{,\delta} - M''u_{\delta} - (H'' + I' + 2K')v_{\delta} + Nu_{\delta}' - (H' + I' + 2M')w_{\delta} + K'v_{\delta}' + (K - M)w_{\delta}',$$

$$C_{255,\delta} = M_{,\delta} - J'u_{\delta} - M''v_{\delta} - (3H'' + 2I'' + 2K'')w_{\delta} + 2Nv_{\delta}' + 2N'w_{\delta}',$$

$$C_{333,\delta} = J_{,\delta} + 3(M - I'u_{\delta}' - I''v_{\delta}),$$

$$C_{334,\delta} = I_{,\delta} + 2K'u_{\delta} + Iv_{\delta} + (J - 2J')u_{\delta}' - 2M''v_{\delta}' - I''w_{\delta}',$$

$$C_{335,\delta} = I_{,\delta} + 2Nu_{\delta} - 2M''u_{\delta} + (J - 2J'')v_{\delta}' + 2w_{\delta} + I'w_{\delta}',$$

$$C_{344,\delta} = J_{,\delta} + Ku_{\delta} + 2K'v_{\delta} - (H - 2I')u_{\delta}' - K''v_{\delta}' - 2M''w_{\delta}',$$

$$C_{345,\delta} = M_{,\delta} + N'u_{\delta} + Nv_{\delta} + (I'' - K'')u_{\delta}' + K'w_{\delta} + (I' - M')v_{\delta}' + (J' - J'')w_{\delta}'.$$

(2.9)
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\[ C_{355,\delta} = J_{,\delta}'' + M u_{\delta} - M' u_{,\delta}' + 2 N w_{\delta} - (H'' - 2 I'') v_{,\delta}' + 2 M'' w_{,\delta}', \]
\[ C_{444,\delta} = H_{,\delta}' + 3 (K v_{,\delta} + J' u_{,\delta}' - K'' w_{,\delta}), \]
\[ C_{455,\delta} = K_{,\delta}'' + 2 N' v_{,\delta}' + 2 M'' u_{,\delta}' + K w_{,\delta} + J v_{,\delta}' + (H' - 2 M') w_{,\delta}', \]
\[ C_{555,\delta} = M_{,\delta}' + M v_{,\delta} + J' u_{,\delta}' + 2 N' w_{,\delta} + 2 M' v_{,\delta}' - (H'' - 2 K'') w_{,\delta}', \]
\[ C_{555,\delta} = H_{,\delta}'' + 3 (M w_{,\delta} + J' v_{,\delta}' + M' w_{,\delta}'). \]
\[ C_{1\beta\gamma\delta} = -C_{\beta\gamma\delta\delta}, \]

where \( H_{,\delta} = L(\tilde{\omega}, H) e_0^k \). From (2.7) and (2.9), we get
\[ C_{222,\delta} + C_{233,\delta} + C_{244,\delta} + C_{255,\delta} = H_{,\delta} + I_{,\delta} + K_{,\delta} + M_{,\delta} = (H + I + K + M)_{,\delta} = (LC)_{,\delta}. \]

(2.10)
\[ C_{322,\delta} + C_{333,\delta} + C_{344,\delta} + C_{355,\delta} = LC u_{,\delta}, \]
\[ C_{222,\delta} + C_{233,\delta} + C_{244,\delta} + C_{255,\delta} = LC v_{,\delta}, \]
\[ C_{225,\delta} + C_{235,\delta} + C_{245,\delta} + C_{255,\delta} = LC w_{,\delta}. \]

From (2.6), it follows that
\[ \hat{L} C_{ijk | h} + LC_{ijk | h} = C_{a(\delta;\beta) e_0^a} e_{\beta j} e_{\gamma k} e_{\delta h}, \]
which implies
\[ \hat{L} C_{ijk | h} = (C_{a(\delta;\beta)} - C_{a(\beta;\delta)}) e_{\alpha a} e_{\beta j} e_{\gamma k} e_{\delta h}. \]

(2.11)
\[ LT_{\mu|k} = (C_{a(\mu;\delta)} + C_{\beta(\delta;\mu)} \delta_{\alpha a} + C_{\alpha(\delta;\beta)} \delta_{\beta j} + C_{a(\delta;\beta)} \delta_{\gamma k}) e_{\alpha a} e_{\beta j} e_{\gamma k} e_{\delta h}. \]

(2.12)
\[ C_{\alpha(\delta;\beta)} - C_{\alpha(\beta;\delta)} = C_{\alpha(\delta;\beta)} \delta_{\alpha a} - C_{\alpha(\beta;\delta)} \delta_{\beta j}. \]

(2.13)

In view of (2.13), equation (2.10) gives
\[ LC u_{,2} = C_{222,2} + C_{233,2} + C_{244,2} + C_{255,2} = C_{222,2} + C_{233,2} + C_{244,2} + C_{255,2} = (LC)_{,2}, \]
\[ LC v_{,2} = C_{222,3} + C_{233,3} + C_{244,3} + C_{255,3} = C_{222,2} + C_{233,2} + C_{244,2} + C_{255,2} = (LC)_{,3}, \]
\[ LC u_{,5} = C_{222,4} + C_{233,4} + C_{244,4} + C_{255,4} = C_{222,2} + C_{233,2} + C_{244,2} + C_{255,2} = (LC)_{,4}, \]
\[ LC v_{,5} = C_{222,5} + C_{233,5} + C_{244,5} + C_{255,5} = C_{222,2} + C_{233,2} + C_{244,2} + C_{255,2} = (LC)_{,5}. \]

(2.14)

Since \( L_{\mu|3} = L(\tilde{\omega}, L) e_3' = LL_{\mu|3} = L_{,3} n_3' = 0 \), \( L_{,4} = L(\tilde{\omega}, L) e_4' = LL_{,4} p'_1 = 0 \) and \( L_{5} = L(\tilde{\omega}, L) e_5' = LL_{,5} q_1' = 0 \), we have

**Proposition 2.2.** The scalar components \( u_2, v_2 \) and \( w_2 \) of the \( v \)-connection vectors \( u_i, v_i \) and \( w_i \) of a five-dimensional Finsler space are given by
\[ u_2 = C^{-1} C_{,3}, \quad v_2 = C^{-1} C_{,4}, \quad w_2 = C^{-1} C_{,5}, \]
and the scalar components \( u_4, u_5, v_3, v_5, w_3, w_5 \) are related by
\[ u_4 = v_3, \quad u_5 = w_3, \quad v_4 = w_3. \]

**III. T-TENSOR OF FORM (A)**

A Finsler space is \( C \)-reducible if and only if the \( T \)-tensor is of the form (A) for \( \rho \neq 0 \). Let \( F^5 \) be a five-dimensional Finsler space with \( T \)-tensor of the form (A). The scalar components of the angular metric tensor \( h_{ij} \) are given by
\[ h_{ij} = (\delta_{ij} - \delta_{(i} \delta_{j)}) e_{\alpha a} e_{\beta j}, \]

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therefore in view of (2.12) and (A), we have
\[ (C_{\alpha\beta\gamma\delta} + C_{\beta\gamma\delta\alpha} + C_{\gamma\delta\alpha\beta} + C_{\alpha\beta\delta\gamma}) = \rho L\{((\delta_{\alpha\beta} - \delta_{\alpha\rho})(\delta_{\gamma\delta} - \delta_{\eta\delta})(\delta_{\eta\delta} - \delta_{\eta\delta})) + ((\delta_{\alpha\rho} - \delta_{\alpha\delta})(\delta_{\gamma\eta} - \delta_{\eta\gamma}))\}, \]
which gives
\[ C_{222,3} = 3\rho L\delta_{2,6}, \quad C_{223,3} = 3\rho L\delta_{2,6}, \quad C_{244,4} = 3\rho L\delta_{2,6}, \quad C_{255,5} = 3\rho L\delta_{2,6}, \]
Putting (3.1) into (2.10), we get
\[ (LC)_{\delta} = 6\rho L\delta_{2,6}, \quad LCU_{\delta} = 6\rho L\delta_{3,6}, \quad LCV_{\delta} = 6\rho L\delta_{4,6}, \quad LCW_{\delta} = 6\rho L\delta_{5,6}. \]
Again from the first equation of (3.1), we get
\[ C_{222,3} = H_{\delta} + 3(J + J' + J'')u_{\delta} + 3(H' + I' + M')v_{\delta} + 3(H'' + I'' + K'')w_{\delta} = 3\rho L\delta_{2,6}. \]
Thus, we have
Theorem 3.1. If the $T$-tensor of a five-dimensional Finsler space is of the form (A), then $\rho$ is given by
\[ \rho = \frac{H_{\delta}}{3L} = \frac{1}{6} C_{3,2} = \frac{1}{6} CU_{3} = \frac{1}{6} CV_{3} = \frac{1}{6} CW_{3}. \]
Theorem 3.2. The scalar components of $\nu$-connection vectors $u_i$ and $v_i$ of a five-dimensional Finsler space with $T$-tensor of the form (A) are given by
\[ u_1 = 0, \quad u_2 = 0, \quad u_3 = C^{-1}C_{2,2}, \quad u_4 = 0, \quad u_5 = 0, \]
\[ v_1 = 0, \quad v_2 = 0, \quad v_3 = 0, \quad v_4 = C^{-1}C_{2,2}, \quad v_5 = 0, \]
\[ w_1 = 0, \quad w_2 = 0, \quad w_3 = 0, \quad w_4 = 0, \quad w_5 = C^{-1}C_{2,2}. \]

IV. $T$–TENSOR OF FORM (B)
Ikeda [8] showed that for an $n$-dimensional Finsler space with $T$-tensor of the form (B)
\[ T_{hijk} = h_{hi}P_{jk} + h_{hj}P_{ik} + h_{hk}P_{ij} + h_{ij}P_{hk} + h_{ik}P_{hj} + h_{jk}P_{hi}, \]
we get
\[ P_{ij} = \frac{1}{n+3} \left( T_{ij} - \frac{T}{2(n+1)}h_{ij} \right), \]
where $T_{ij} = T_{hijk} g^{hk}$ and $T = T_{ij} g^{ij}$. Therefore (B) becomes
\[ T_{hijk} = \frac{1}{n+3} \left( h_{hi}T_{jk} + h_{hj}T_{ik} + h_{hk}T_{ij} + h_{ij}T_{hk} + h_{ik}T_{hj} + h_{jk}T_{hi} \right) \]
\[ - \frac{T}{(n+1)(n+3)}(h_{hi}h_{jk} + h_{hj}h_{ik} + h_{hk}h_{ij}). \]
Thus, for a five-dimensional Finsler space, we have
\[ T_{hijk} = \frac{1}{8} \left( h_{hi}T_{jk} + h_{hj}T_{ik} + h_{hk}T_{ij} + h_{ij}T_{hk} + h_{ik}T_{hj} + h_{jk}T_{hi} \right) - \frac{T}{6}(h_{hi}h_{jk} + h_{hj}h_{ik} + h_{hk}h_{ij}). \]
Let $T_{\alpha\delta}$ be the scalar components of $LT_{hijk}$, i.e.,
\[ LT_{hi} = T_{\alpha\delta} e_{\alpha\beta} e_{\beta i}. \]
In view of (2.12) and (4.1), we get
(C_{\alpha \beta \delta} \alpha + C_{\beta \delta \alpha} \beta + C_{\alpha \delta \beta} \gamma + C_{\alpha \beta \delta} \gamma) = \frac{1}{8}[((\delta_{\alpha \beta} - \delta_{\alpha \delta} \delta_{\beta}) T_{\beta \delta} + (\delta_{\alpha \beta} - \delta_{\alpha \gamma} \delta_{\beta} \gamma)) T_{\beta \delta} + (\delta_{\alpha \gamma} - \delta_{\alpha \delta} \delta_{\gamma}) T_{\beta \delta} + (\delta_{\alpha \beta} - \delta_{\alpha \gamma} \delta_{\beta} \gamma)) T_{\beta \delta}]
+(\delta_{\alpha \delta} - \delta_{\alpha \gamma} \delta_{\beta} \gamma) T_{\beta \delta} + (\delta_{\alpha \beta} - \delta_{\alpha \gamma} \delta_{\beta} \gamma) T_{\beta \delta} + (\delta_{\alpha \gamma} - \delta_{\alpha \delta} \delta_{\gamma}) T_{\beta \delta} + (\delta_{\alpha \beta} - \delta_{\alpha \gamma} \delta_{\beta} \gamma)) T_{\beta \delta}
- \frac{LT}{6}[(\delta_{\alpha \beta} - \delta_{\alpha \gamma} \delta_{\beta} \gamma) (\delta_{\alpha \delta} - \delta_{\alpha \gamma} \delta_{\beta} \gamma) (\delta_{\alpha \gamma} - \delta_{\alpha \delta} \delta_{\gamma}) + (\delta_{\alpha \beta} - \delta_{\alpha \gamma} \delta_{\beta} \gamma) (\delta_{\alpha \delta} - \delta_{\alpha \gamma} \delta_{\beta} \gamma) + (\delta_{\alpha \gamma} - \delta_{\alpha \delta} \delta_{\gamma}) (\delta_{\alpha \beta} - \delta_{\alpha \gamma} \delta_{\beta} \gamma)],

which gives

\begin{align*}
C_{222,\delta} &= \frac{1}{8}[3T_{2,\delta} + 3T_{2,2,\delta} - \frac{1}{2} LT \delta_{2,\delta}], \\
C_{233,\delta} &= \frac{1}{8}[T_{3,3,\delta} + T_{2,3,\delta} + 2T_{2,4,\delta} - \frac{1}{6} LT \delta_{2,3,\delta}], \\
C_{244,\delta} &= \frac{1}{8}[T_{4,4,\delta} + T_{2,4,\delta} + 2T_{2,5,\delta} - \frac{1}{6} LT \delta_{2,4,\delta}], \\
C_{255,\delta} &= \frac{1}{8}[T_{5,5,\delta} + T_{2,5,\delta} + 2T_{2,6,\delta} - \frac{1}{6} LT \delta_{2,5,\delta}], \\
C_{322,\delta} &= \frac{1}{8}[T_{2,2,\delta} + T_{3,3,\delta} + 2T_{3,4,\delta} - \frac{1}{6} LT \delta_{3,2,\delta}], \\
C_{333,\delta} &= \frac{1}{8}[3T_{3,3,\delta} + 3T_{3,3,\delta} - \frac{1}{2} LT \delta_{3,3,\delta}], \\
C_{344,\delta} &= \frac{1}{8}[T_{4,4,\delta} + T_{3,4,\delta} + 2T_{3,5,\delta} - \frac{1}{6} LT \delta_{3,4,\delta}], \\
C_{355,\delta} &= \frac{1}{8}[T_{5,5,\delta} + T_{3,5,\delta} + 2T_{3,6,\delta} - \frac{1}{6} LT \delta_{3,5,\delta}], \\
C_{224,\delta} &= \frac{1}{8}[T_{2,2,\delta} + T_{4,4,\delta} + 2T_{4,5,\delta} - \frac{1}{6} LT \delta_{2,4,\delta}], \\
C_{225,\delta} &= \frac{1}{8}[T_{2,2,\delta} + T_{5,5,\delta} + 2T_{5,6,\delta} - \frac{1}{6} LT \delta_{2,5,\delta}], \\
C_{444,\delta} &= \frac{1}{8}[3T_{4,4,\delta} + 3T_{4,4,\delta} - \frac{1}{2} LT \delta_{4,4,\delta}], \\
C_{334,\delta} &= \frac{1}{8}[T_{4,4,\delta} + 2T_{3,4,\delta} + T_{3,3,\delta} - \frac{1}{6} LT \delta_{4,3,\delta}], \\
C_{335,\delta} &= \frac{1}{8}[T_{5,5,\delta} + 2T_{3,5,\delta} + T_{3,3,\delta} - \frac{1}{6} LT \delta_{5,3,\delta}], \\
C_{455,\delta} &= \frac{1}{8}[T_{4,4,\delta} + 2T_{4,5,\delta} + T_{5,5,\delta} - \frac{1}{6} LT \delta_{4,5,\delta}], \\
C_{555,\delta} &= \frac{1}{8}[3T_{5,5,\delta} + 3T_{5,5,\delta} - \frac{1}{2} LT \delta_{5,5,\delta}], \\
C_{445,\delta} &= \frac{1}{8}[T_{5,5,\delta} + 2T_{4,5,\delta} + T_{4,4,\delta} - \frac{1}{6} LT \delta_{5,5,\delta}].
\end{align*}

Putting (4.2) into (2.10), we get
(LC)_{,\delta} = \frac{1}{8} \{ 6T_{2\delta} + (3T_{22} + T_{33} + T_{44} + T_{55})\delta_{2\delta} + 2T_{23}\delta_{3\delta} + 2T_{24}\delta_{4\delta} + 2T_{25}\delta_{5\delta} - LT\delta_{2\delta} \},

LCu_{,\delta} = \frac{1}{8} \{ 6T_{3\delta} + (3T_{22} + T_{23} + T_{44} + T_{55})\delta_{3\delta} + 2T_{25}\delta_{2\delta} + 2T_{34}\delta_{4\delta} + 2T_{35}\delta_{5\delta} - LT\delta_{3\delta} \},

LCv_{,\delta} = \frac{1}{8} \{ 6T_{4\delta} + (3T_{22} + T_{23} + T_{33} + T_{55})\delta_{4\delta} + 2T_{24}\delta_{2\delta} + 2T_{34}\delta_{3\delta} + 2T_{45}\delta_{5\delta} - LT\delta_{4\delta} \},

LCw_{,\delta} = \frac{1}{8} \{ 6T_{5\delta} + (T_{22} + T_{33} + T_{44} + T_{55})\delta_{5\delta} + 2T_{25}\delta_{2\delta} + 2T_{35}\delta_{3\delta} + 2T_{45}\delta_{4\delta} - LT\delta_{5\delta} \}.

Therefore,

(LC)_{,2} = \frac{1}{8} \{ 9T_{22} + T_{33} + T_{44} + T_{55} - LT \},

(LC)_{,3} = T_{23},

(LC)_{,4} = T_{24},

(LC)_{,5} = T_{25},

LCu_{,3} = T_{23},

LCu_{,4} = \frac{1}{8} \{ T_{22} + 9T_{33} + T_{44} + T_{55} - LT \},

LCu_{,4} = T_{23},

LCu_{,5} = T_{35},

LCu_{,5} = T_{35},

LCv_{,3} = T_{25},

LCv_{,4} = T_{25},

LCv_{,5} = T_{45},

LCw_{,3} = T_{35},

LCw_{,4} = T_{35},

LCw_{,5} = T_{45},

LCw_{,5} = T_{45},

LT = T_{\alpha\beta\gamma\delta} = T_{\alpha\alpha} = T_{22} + T_{33} + T_{44} + T_{55}.\}

Thus, in view of (4.3), we have

Theorem 4.1. If the T–tensor of a five-dimensional Finsler space is of the form (B), the scalar components of the tensor T_{ij} are given by

T_{1\alpha} = 0,

T_{22} = (LC)_{,2},

T_{33} = LCu_{,3},

T_{44} = LCv_{,4},

T_{55} = LCw_{,5},

T_{23} = LCu_{,2} = (LC)_{,3},

T_{24} = LCv_{,2} = (LC)_{,4},

T_{25} = LCw_{,2} = (LC)_{,5},

T_{34} = LCu_{,4} = LCv_{,4},

T_{35} = LCu_{,5} = LCw_{,5},

T_{45} = LCw_{,4} = LCv_{,5},

and

T = C_{,2} + Cu_{,5} + Cv_{,4} + CW_{,5}.

V. T–TENSOR OF FORM (C)

U. P. Singh et al. [4] showed that the T–tensor of a C–2 like Finsler space is of the form (C)

T_{hjk} = \rho C_{h} C_{j} C_{k} + a_{h} C_{h} C_{j} C_{k} + a_{h} C_{k} C_{j} C_{k} + a_{j} C_{h} C_{j} C_{k} + a_{k} C_{h} C_{j} C_{j}.

Let a_{a} be the scalar components of La_{i}, i.e.,

La_{i} = a_{a} \epsilon^{a}_{ij}.\}

Since e_{2\alpha} = C_{j}, we get C_{j} = C_{j} \epsilon_{2\alpha}^{a} \epsilon^{a}_{ij}.

Therefore in view of (2.12) and (C), we have

(C_{\alpha\beta\gamma\delta} + C_{\beta\alpha\delta} \epsilon^{a}_{\alpha} + C_{\gamma\alpha\delta} \epsilon^{a}_{\beta} + C_{\alpha\beta\gamma} \epsilon^{a}_{\delta}) = \rho LC^{2} \delta_{\alpha}^{\beta} \delta_{\beta}^{\gamma} \delta_{\gamma}^{\delta} + C^{3}(a_{\alpha} \delta_{\beta}^{\gamma} \delta_{\gamma}^{\delta} + a_{\beta} \delta_{\alpha}^{\gamma} \delta_{\gamma}^{\delta} + a_{\gamma} \delta_{\alpha}^{\beta} \delta_{\beta}^{\delta} + a_{\delta} \delta_{\alpha}^{\beta} \delta_{\beta}^{\gamma}) \)

which gives
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$C_{222,\delta} = C^3(\rho LC + 3a_2)\delta_{2,\delta} + C^3a_4\delta_{2,\delta}, \quad C_{233,\delta} = 0, \quad C_{244,\delta} = 0, \quad C_{255,\delta} = 0,$
$C_{322,\delta} = C^3a_4\delta_{2,\delta}, \quad C_{333,\delta} = 0, \quad C_{344,\delta} = 0, \quad C_{355,\delta} = 0,$
$C_{224,\delta} = C^3a_3\delta_{2,\delta}, \quad C_{225,\delta} = C^3a_3\delta_{2,\delta}, \quad C_{244,\delta} = 0, \quad C_{334,\delta} = 0,$
$C_{335,\delta} = 0, \quad C_{445,\delta} = 0, \quad C_{455,\delta} = 0.$

Putting (5.1) into (2.10), we get

$$(LC)_{\delta} = C^3(\rho LC + 3a_2)\delta_{2,\delta} + C^3a_3\delta_{2,\delta}, \quad LCu_{\delta} = C^3a_3\delta_{2,\delta}, \quad LCv_{\delta} = C^3a_3\delta_{2,\delta}, \quad LCw_{\delta} = C^3a_3\delta_{2,\delta}.$$  

Since $T_{hijk}$ is an indicatory tensor, from (C) it follows that $a_i = a_iy' = 0$. Thus, we have:

**Theorem 5.1.** If the $T$–tensor of a five-dimensional Finsler space is of the form (C), the scalar components $a_a$ of the $La_i$ are given by

$a_1 = 0, \quad a_2 = \frac{L}{4}(C^{-3}C_{2,\delta} - \rho C), \quad a_3 = LC^{-1}u_2 = C^{-3}(LC)_{,3}, \quad a_4 = LC^{-2}v_2 = C^{-3}(LC)_{,3}, \quad a_5 = LC^{-2}w_2 = C^{-3}(LC)_{,5}.$

**Theorem 5.2.** In a five-dimensional Finsler space with $T$–tensor of the form (C), the scalar components of $v$–connection vectors $u_i, v_i$ and $w_i$ vanish if the scalar components $a_3, a_4$ and $a_5$ of $La_i$ vanish.

**Corollary 5.1.** In a five-dimensional Finsler space with $T$–tensor of the form (C), the $v$–connection vectors $u_i, v_i$ and $w_i$ vanish if the scalar components $a_3, a_4$ and $a_5$ of $La_i$ vanish.

**VI. T–2 LIKE FINSLER SPACE**

A non-Riemannian Finsler space $F^n (n > 2)$ is called $T–2$ like Finsler space if the $T$–tensor $T_{hijk}$ is written in the form

$$T_{hijk} = \rho C_{h}C_{i}C_{j}C_{k}. \quad (6.1)$$

**Theorem 6.1.** In a $T–2$ like five-dimensional Finsler space, the $v$–connection vectors $u_i, v_i$ and $w_i$ vanish.

**Theorem 6.2.** In a $T–2$ like five-dimensional Finsler space, $\rho$ is given by

$$\rho = C^{-3}C_{2}.$$  

**REFERENCES**

