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A Note on the Schnute Growth Model

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Abstract:- Mathematical models of growth have been developed a long period of time. Estimating the lag time in the growth process is a practically important problem. Any sigmoidal function can be good illustration for the concept of lag time. The Schnute growth model is described by free parameters, each contributing to the characteristics of the curve: an initial lag or period of slow growth; a period of rapid exponential growth; a period of reduced growth rate. In this note we provide estimates for the one–sided Hausdorff approximation of the step–function by sigmoidal Schnute function - ($t_{new-lag}$). Numerical examples, illustrating our results are given, too.

Keywords:- Sigmoidal Schnute function, Step function, Hausdorff distance, Lag time.

I. INTRODUCTION

Growth curves are found in a wide range of disciplines, such as biology, chemistry and medical science. Estimating the lag time in the growth process is a practically important problem [1], [2].
Any sigmoidal function can be good illustration for the concept of lag time. The growth model is described by free parameters, each contributing to the characteristics of the sigmoidal function.
These parameters may be useful for describing biologically relevant metrics as a lag phase, the growth phase, and the plateau phase.

The lag time - t_{lag} (see Fig. 1) is estimated by extending the tangent at inflection point to the initial baseline.
The Schnute curve is described by free parameters, each contributing to the characteristics of the curve: an initial lag or period of slow growth; a period of rapid exponential growth; a period of reduced growth rate.
The Schnute function finds applications in many scientific fields, including population dynamics, bacterial growth, population ecology, plant biology, chemistry and statistics.
In his classical paper, Schnute [3] considered the accelerated growth rate of species, and solved the model system:

$$\begin{aligned} \frac{dL}{dt} &= Lk, \\ \frac{dk}{dt} &= -k(a+bk), \end{aligned} \qquad (1)$$

where parameters a and b are any constants. Evidently

$$\begin{aligned} \frac{d^2L}{dt^2} &= \frac{dL}{dt}k + L\frac{dk}{dt} \\ &= \frac{dL}{dt}k - Lk(a+bk) \\ &= \frac{dL}{dt}k - \frac{dL}{dt}(a+bk) \\ &= \frac{dL}{dt}\left(-a+(1-b)k\right). \end{aligned} \qquad (2)$$

The basic form of this model is:

$$L(t;l_1,l_2,a,b)=\left(l_1^b+\left(l_2^b-l_1^b\right)\frac{1-e^{-a(t-t_1)}}{1-e^{-a(t_2-t_1)}}\right)^{\frac{1}{b}}. \qquad (3)$$

The values t_1 and t_2 are fixed and are normally taken to be the smallest and largest diameters in the data. $l_1=L(t_1)$ and $l_2=L(t_2)$ are the initial and final population densities, respectively (generally $l_2>l_1$); $a\neq 0$ and $b\neq 0$ are rate parameters.

For some modelling aspects and parameter estimations, see [4], [6], [7], [8], [9], [14], [15], [16], [17], [18], [19], [20], [21].
At $t\to\infty$ the function arrived an asymptotic value of [14], [22], [23]

$$L_\infty=\left(\frac{l_2^b e^{at_2}-l_1^b e^{at_1}}{e^{at_2}-e^{at_1}}\right)^{\frac{1}{b}}. \qquad (4)$$

The coordinate of the inflection point, will be obtained from

$$t_s=t_1+t_2-\frac{1}{a}\ln\left(\frac{b\left(e^{at_2}l_2^b-e^{at_1}l_1^b\right)}{l_2^b-l_1^b}\right) \qquad (5)$$

and the corresponding ordinate is

$$l_s=L(t_s)=\left(\frac{(1-b)\left(e^{at_2}l_2^b-e^{at_1}l_1^b\right)}{e^{at_2}-e^{at_1}}\right)^{\frac{1}{b}}. \qquad (6)$$

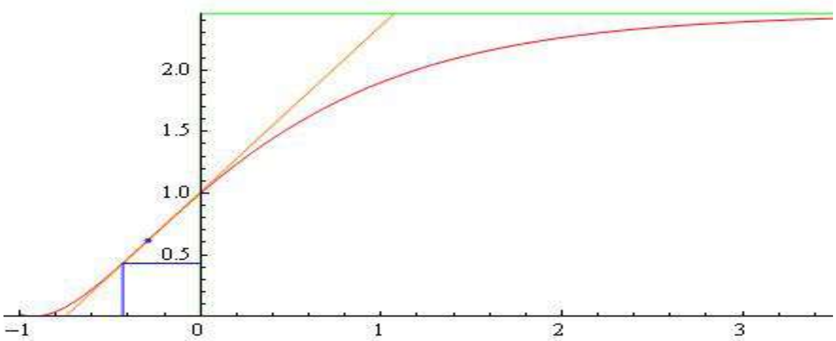

Figure 1: Definitions: a) t_{lag} - is estimated by extending the tangent at inflection point to the initial baseline; b) $t_{new-lag}$ - the one–sided Hausdorff approximation – d of the step–function by sigmoidal Schnute function. The parameters are: $a=1.1$, $b=0.5$, $t_1=0$, $t_2=1$, $l_1=1$, $l_2=1.9$; $d=0.430683$.

In this note we prove estimates for the one–sided Hausdorff approximation of the interval step–function by sigmoidal Schnute function - ($t_{new-lag}$).
Let us point out that Hausdorff distance is the most natural measuring criteria for the approximation of bounded discontinuous function [24], [25].

II. PRELIMINARIES

Definition 1. Define the shifted step function h_{t_1} as:

$$h_{t_1}(t) = \begin{cases} 0, & \text{if} \quad t < t_1, \\ [0, L_\infty] & \text{if} \quad t = t_1, \\ L_\infty, & \text{if} \quad t > t_1. \end{cases} \qquad (7)$$

Definition 2. The Hausdorff distance (H-distance) $\rho(f,g)$ between two interval functions f,g on $\Omega \subseteq \mathrm{P}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathrm{P}$ [26], [27]. More precisely,

$$\rho(f,g) = \max\{ \sup_{A \in F(f)} \inf_{B \in F(g)} \| A - B \|, \sup_{B \in F(g)} \inf_{A \in F(f)} \| A - B \| \}, \qquad (8)$$

wherein $\|.\|$ is any norm in P^2, e. g. the maximum norm $\|(t,x)\| = \max\{|t|,|x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in P^2 is $\| A - B \| = max(|t_A - t_B|, |x_A - x_B|)$.

III. MAIN RESULTS

We study the one–sided Hausdorff approximation of the shifted step function $h_{t_1}(t)$ by sigmoidal Schnute function $L(t)$.

The following Theorem is valid

Theorem 3.1 *For the one–sided Hausdorff distance d between the function $h_{t_1}(t)$ and the Schnute function (3) the following holds:*

$$d \approx d^* = \frac{l_1}{1 + \dfrac{a l_1^{1-b}(l_2^b - l_1^b)}{b(1 - e^{-a(t_2 - t_1)})}}. \qquad (9)$$

Proof. The one–sided Hausdorff distance d satisfies the relation (see, Figure 1)

$$L(t_1 - d) = \left(l_1^b + \left(l_2^b - l_1^b\right) \frac{1 - e^{ad}}{1 - e^{-a(t_2 - t_1)}} \right)^{\frac{1}{b}} = d. \qquad (10)$$

Let us examine the function

$$F(d) = \left(l_1^b + \left(l_2^b - l_1^b\right) \frac{1 - e^{ad}}{1 - e^{-a(t_2 - t_1)}} \right)^{\frac{1}{b}} - d. \qquad (11)$$

Consider function

$$G(d) = l_1 - \left(1 + \frac{a l_1^{1-b}(l_2^b - l_1^b)}{b(1 - e^{-a(t_2 - t_1)})} \right) d. \qquad (12)$$

From Taylor expansion

$$\left(l_1^b + \left(l_2^b - l_1^b\right) \frac{1 - e^{ad}}{1 - e^{-a(t_2 - t_1)}} \right)^{\frac{1}{b}} - d = l_1 - \left(1 + \frac{a l_1^{1-b}(l_2^b - l_1^b)}{b(1 - e^{-a(t_2 - t_1)})} \right) d + O(d^2)$$

we obtain $G(d) - F(d) = O(d^2)$ (see, Fig. 2).

This completes the proof of the theorem.

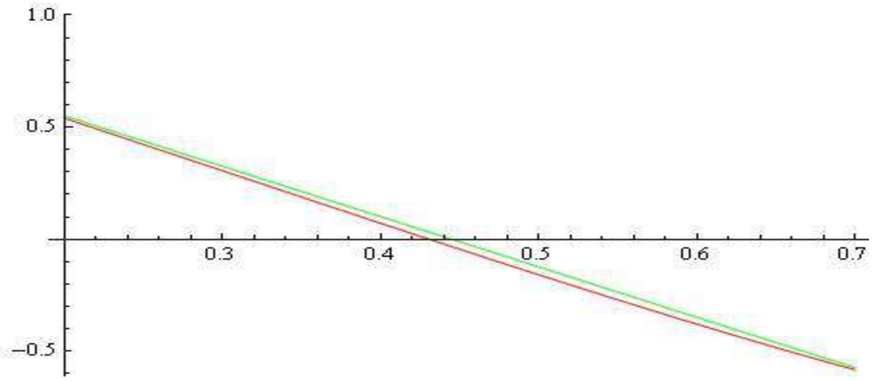

Figure 2: The functions $F(d)$ and $G(d)$ for $a=1.1$, $b=0.5$, $t_1=0$, $t_2=1$, $l_1=1$, $l_2=1.9$.

The bound for d computed by nonlinear equation (11) is $d=0.430683$. From (9) we have $d \approx d^* = 0.44486$.
The ”new” lag time is then given in terms of the one–sided Hausdorff distance - d.

Year	*Weight*	*The appropriate fitting by Schnute function* (3)
4	11.78	10.2742
5	18.43	18.43
6	25.21	25.5064
7	30.78	30.5749
8	33.03	33.8333
9	35.66	35.8046
10	36.96	36.9573
11	37.97	37.6186
12	38.04	37.9939
13	39.20	38.2056
14	36.50	38.3247
15	37.21	38.3915
16	39.97	38.429
17	38.45	38.45

Table 1: The oil palm yield data [28], [29]

IV. COMPUTATIONAL ISSUES. FITTING THE NONLINEAR SCHNUTE GROWTH MODEL AGAINST EXPERIMENTAL OIL PALM DATA [28], [29]

Simple module in *CAS Mathematica* for calculation of the value of the one–sided Hausdorff distance d between the shifted step function and the sigmoidal Schnute function is visualized on Figure 3.

```
a = Input[" a"];(=1.1 =)
Print[" a = ", a];
b = Input[" b"];(=0.5 =)
Print[" b = ", b];
t1 = Input[" t1"];(=0 =)
Print[" t1 = ", t1];
t2 = Input[" t2"];(=1 =)
Print[" t2 = ", t2];
l1 = Input[" l1"];(=1 =)
Print[" l1 = ", l1];
l2 = Input[" l2"];(=1.9 =)
Print[" l2 = ", l2];
Print["The following nonlinear equation is used to determination of
the one-sided Hausdorff distance between shifted step function and
Schnute growth curve - d (the new_lag_time): "];
FindRoot[(l1^b + (l2^b - l1^b) * (1 - Exp[a * d]) / (1 - Exp[-a * (t2 - t1)]))^(1 / b) - d, {d, 0}]
  a = 1.1
  b = 0.5
  t1 = 0
  t2 = 1
  l1 = 1
  l2 = 1.9
The following nonlinear equation is used to determination of
the one-sided Hausdorff distance between Heaviside function and
Schnute growth curve - d (the new_lag_time):
 {d → 0.430683}
```

Figure 3: Simple module implemented in programming environment *CAS Mathematica* for calculation of the value of the one–sided Hausdorff distance d between the shifted step function and the sigmoidal Schnute function.

Example:. The oil palm yield growth data is given in Table 1.
The appropriate fitting of the experimental data by the Schnute growth function $L(t)$ with $a = 0.58$, $b = 0.015$, $t_1 = 5$, $t_2 = 17$, $l_1 = 18.43$, $l_2 = 38.45$ is visualized on Figure 4 (see also the last column of Table 1).

V. CONCLUSION REMARKS

We propose a modified new–lag–time in terms of Hausdorff distance - d.
On a number of computational examples we demonstrate the applicability of the Schnute growth function to approximate the step function and consequently to be employed in fitting time course experimental data related to population dynamics.

Several sigmoidal functions (logistic [30],[31], Gompertz [32], Richards [33], [34], [35], [36], Chapman–Richards (based on the Von Bertalanffy's approach [37]), and Stannard [38]) were compared to describe a growth curve.

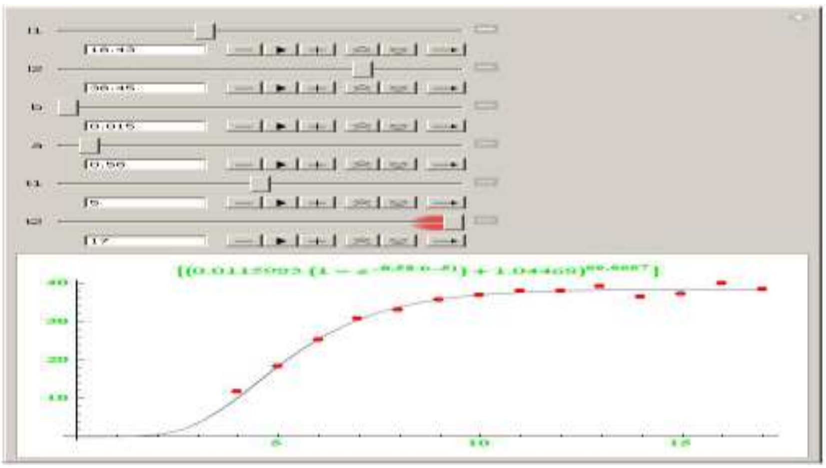

Figure 4: The appropriate fitting of experimental data by the Schnute growth function $L(t)$ with $a = 0.58$, $b = 0.015$, $t_1 = 5$, $t_2 = 17$, $l_1 = 18.43$, $l_2 = 38.45$.

For some modelling aspects and parameter estimations, see [18], [1], [2], [39], [40], [41], [42].

The Hausdorff approximation of the interval step function by the logistic and other sigmoid functions is discussed from various approximation, computational and modelling aspects in [43]–[51].

Definition 3. Define the modified Schnute growth function $L^*(t)$ as ([14], [15]):

$$L^*(t;\mu,\alpha,\beta,\lambda) = \left(\mu\frac{1-\beta}{\alpha}\right) \cdot \left(\frac{1-\beta e^{\alpha\lambda+1-\beta-\alpha t}}{1-\beta}\right)^{\frac{1}{\beta}} \qquad (13)$$

where μ, α, β and λ are growth parameters.

The appropriate fitting of the experimental data by the modified Schnute growth function $L^*(t)$ with $\alpha = 0.442$, $\beta = 0.305$, $\lambda = 2.44$ and $\mu = 7.59$ is visualized on Figure 5.

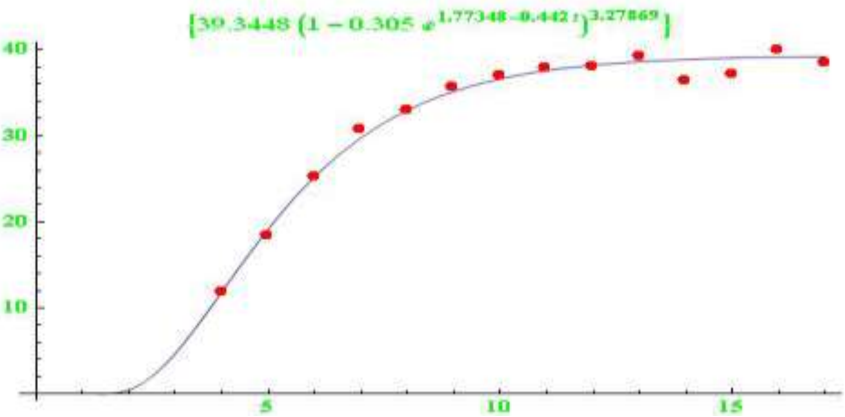

Figure 5: The appropriate fitting of experimental data by the modified Schnute growth function $L^*(t)$ with $\alpha = 0.442$, $\beta = 0.305$, $\lambda = 2.44$ and $\mu = 7.59$.

Based on the methodology proposed in present note, the reader may be formulate the corresponding approximation problems on his/her own.

For some comparisons and selections of growth models using the Schnute model, see [10], [11], [5], [12] and [14].

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