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Zeros of Polynomials in Ring-shaped Regions

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Abstract:- In this paper we subject the coefficients of a polynomial and their real and imaginary parts to certain conditions and give bounds for the number of zeros in a ring-shaped region. Our results generalize many previously known results and imply a number of new results as well.

Mathematics Subject Classification: 30 C 10, 30 C 15
Keywords and Phrases:- Coefficient, Polynomial, Zero

I. INTRODUCTION AND STATEMENT OF RESULTS

A large number of research papers have been published so far on the location in the complex plane of some or all of the zeros of a polynomial in terms of the coefficients of the polynomial or their real and imaginary parts. The famous Enestrom-Kakeya Theorem [5] states that if the coefficients of the polynomial $P(z)=\sum_{j=0}^{n}a_j z^j$ satisfy $0 \le a_0 \le a_1 \le ...... \le a_{n-1} \le a_n$, then all the zeros of P(z) lie in the closed disk $|z| \le 1$.

By putting a restriction on the coefficients of a polynomial similar to that of the Enestrom-Kakeya Theorem, Mohammad [6] proved the following result:

Theorem A: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial such that $0 < a_0 \le a_1 \le ...... \le a_{n-1} \le a_n$. Then the number of zeros of P(z) in $|z| \le \frac{1}{2}$ does not exceed

$$1+\frac{1}{\log 2}\log\frac{a_n}{a_0}.$$

For polynomials with complex coefficients, Dewan [1] proved the following results:

Theorem B: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial such that

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2},\ j=0,1,2,.....,n,$$

for some real α and β and

$$0 < |a_0| \le |a_1| \le ...... \le |a_{n-1}| \le |a_n|.$$

Then the number of zeros of P(z) in $|z| \le \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2}\log\frac{|a_n|(\cos\alpha+\sin\alpha+1)+2\sin\alpha\sum_{j=0}^{n-1}|a_j|}{|a_0|}.$$

Theorem C: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ b e a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j$, $j=0,1,2,......,n$ such that

$$0 < \alpha_0 \le \alpha_1 \le ...... \le \alpha_{n-1} \le \alpha_n.$$

Then the number of zeros of P(z) in $|z| \le \frac{1}{2}$ does not exceed

$$1+\frac{1}{\log 2}\log\frac{\alpha_n+\sum_{j=0}^{n}|\beta_j|}{|a_0|}.$$

Recently Gulzar [3,4] proved the following results:

Theorem D: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k \ge 1, o<\tau \le 1$ and for some integer $\lambda, 0 \le \lambda \le n-1$,

$$\alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_{\lambda+1} \ge k\alpha_\lambda \ge \alpha_{\lambda-1} \ge ...... \ge \alpha_1 \ge \tau\alpha_0.$$

Then P(z) has no zero in $|z| < \frac{|a_0|}{M_1}$, where

$$M_1 = |a_n| + \alpha_n + 2(k-1)|\alpha_\lambda| - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n}|\beta_j| .$$

Theorem E: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some real $\alpha, \beta; |\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}, j=0,1,2,......,n$ and for some $k \ge 1, o<\tau \le 1$ and some integer $\lambda, 0 \le \lambda \le n-1$,

$$|a_n| \ge |a_{n-1}| \ge ...... \ge |a_{\lambda+1}| \ge k|a_\lambda| \ge |a_{\lambda-1}| \ge ...... \ge |a_1| \ge \tau|a_0| .$$

Then P(z) has no zeros in $|z| < \frac{|a_0|}{M_2}$, where

$$M_2 = |a_n|(\cos\alpha + \sin\alpha + 1) + 2|a_\lambda|(k + k\sin\alpha - 1) - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0|$$
$$+ 2\sin\alpha \sum_{j=1, j\ne\lambda}^{n-1}|a_j|.$$

Theorem F: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j, j=0,1,......,n$ such that for some $k_1 \ge 1, k_2 \ge 1, 0<\tau \le 1$,

$$k_1\alpha_n \ge k_2\alpha_{n-1} \ge \alpha_{n-2} ......\alpha_1 \ge \tau\alpha_0$$

Then P(z) has no zero in $|z| < \frac{|a_0|}{M_3}$, where

$$M_3 = |a_n| + (k_1|\alpha_n| + k_2|\alpha_{n-1}|) + (k_1\alpha_n - k_2\alpha_{n-1}) + \alpha_{n-1} - (|\alpha_n| + |\alpha_{n-1}|)$$
$$- \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n-1}|\beta_j|.$$

Theorem G: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k_1 \ge 1, k_2 \ge 1, 0<\tau \le 1$,

$$k_1|a_n| \ge k_2|a_{n-1}| \ge |a_{n-2}| ...... \ge |a_1| \ge \tau|a_0| .$$

Then P(z) has no zero in $|z| < \frac{|a_0|}{M_4}$, where

$$M_4 = k_1|a_n|(\cos\alpha + \sin\alpha + 1) + k_2|a_{n-1}|(\sin\alpha - \cos\alpha + 1) - |a_{n-1}|(1 - \cos\alpha)$$
$$- \tau|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha\sum_{j=1}^{n-2}|a_j|.$$

In this paper, we prove the following results:

Theorem 1: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k \geq 1, o < \tau \leq 1$ and for some integer $\lambda, 0 \leq \lambda \leq n-1$,

$$\alpha_n \geq \alpha_{n-1} \geq ...... \geq \alpha_{\lambda+1} \geq k\alpha_\lambda \geq \alpha_{\lambda-1} \geq ...... \geq \alpha_1 \geq \tau\alpha_0.$$

Then the number of zeros of P(z) in $|z| \leq \frac{R}{c} (R > 0, c > 1)$ does not exceed

$$\frac{1}{\log c}\log\frac{|a_n|R^{n+1} + |a_0| + R^n[\alpha_n + 2(k-1)|\alpha_\lambda| - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n}|\beta_j|]}{|a_0|}$$

for $R \geq 1$ and

$$\frac{1}{\log c}\log\frac{|a_n|R^{n+1} + |a_0| + R[\alpha_n + 2(k-1)|\alpha_\lambda| - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n}|\beta_j|]}{|a_0|}$$

for $R \leq 1$.

Combining Theorem 1 and Theorem D, we get a bound for the number of zeros of P(z) in a ring-shaped region as follows:

Corollary 1: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k \geq 1, o < \tau \leq 1$ and for some integer $\lambda, 0 \leq \lambda \leq n-1$,

$$\alpha_n \geq \alpha_{n-1} \geq ...... \geq \alpha_{\lambda+1} \geq k\alpha_\lambda \geq \alpha_{\lambda-1} \geq ...... \geq \alpha_1 \geq \tau\alpha_0.$$

Then the number of zeros of P(z) in $\frac{|a_0|}{M_1} \leq |z| \leq \frac{R}{c} (R > 0, c > 1)$ does not exceed

$$\frac{1}{\log c}\log\frac{|a_n|R^{n+1} + |a_0| + R^n[\alpha_n + 2(k-1)|\alpha_\lambda| - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n}|\beta_j|]}{|a_0|}$$

for $R \geq 1$ and

$$\frac{1}{\log c}\log\frac{|a_n|R^{n+1} + |a_0| + R[\alpha_n + 2(k-1)|\alpha_\lambda| - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n}|\beta_j|]}{|a_0|}$$

for $R \leq 1$, where

$$M_1 = |a_n| + \alpha_n + 2(k-1)|\alpha_\lambda| - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n}|\beta_j|.$$

Theorem 2: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some real $\alpha,\beta;\left|\arg a_j-\beta\right|\le\alpha\le\frac{\pi}{2},\ j=0,1,2,......,n$ and for some $k\ge1, o<\tau\le1$ and some integer $\lambda, 0\le\lambda\le n-1$,

$$|a_n|\ge|a_{n-1}|\ge......\ge|a_{\lambda+1}|\ge k|a_\lambda|\ge|a_{\lambda-1}|\ge......\ge|a_1|\ge\tau|a_0|.$$

Then the number of zeros of P(z) in $|z|\le\frac{R}{c}\,(R>0,c>1)$ does not exceed

$$\frac{1}{\log c}\log\frac{1}{|a_0|}\left[\begin{array}{l}|a_n|R^{n+1}+|a_0|+R^n\{|a_n|(\cos\alpha+\sin\alpha)+2|a_\lambda|(k+k\sin\alpha-1)\\ -\tau|a_0|(\cos\alpha-\sin\alpha+1)+|a_0|\ +2\sin\alpha\displaystyle\sum_{j=1,\,j\ne\lambda}^{n-1}|a_j|\}\end{array}\right]\quad\text{for } R\ge1$$

and

$$\frac{1}{\log c}\log\frac{1}{|a_0|}\left[\begin{array}{l}|a_n|R^{n+1}+|a_0|+R^n\{|a_n|(\cos\alpha+\sin\alpha)+2|a_\lambda|(k+k\sin\alpha-1)\\ -\tau|a_0|(\cos\alpha-\sin\alpha+1)+|a_0|\ +2\sin\alpha\displaystyle\sum_{j=1,\,j\ne\lambda}^{n-1}|a_j|\}\end{array}\right]\quad\text{for } R\le1.$$

Combining Theorem 2 and Theorem E, we get a bound for the number of zeros of P(z) in a ring-shaped region as follows:

Corollary 2: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n such that for some

real $\alpha,\beta;\left|\arg a_j-\beta\right|\le\alpha\le\frac{\pi}{2},\ j=0,1,2,......,n$ and for some $k\ge1, o<\tau\le1$ and some integer $\lambda, 0\le\lambda\le n-1$,

$$|a_n|\ge|a_{n-1}|\ge......\ge|a_{\lambda+1}|\ge k|a_\lambda|\ge|a_{\lambda-1}|\ge......\ge|a_1|\ge\tau|a_0|.$$

Then the number of zeros of P(z) in $\frac{|a_0|}{M_2}\le|z|\le\frac{R}{c}\,(R>0,c>1)$ does not exceed

$$\frac{1}{\log c}\log\frac{1}{|a_0|}\left[\begin{array}{l}|a_n|R^{n+1}+|a_0|+R^n\{|a_n|(\cos\alpha+\sin\alpha)+2|a_\lambda|(k+k\sin\alpha-1)\\ -\tau|a_0|(\cos\alpha-\sin\alpha+1)+|a_0|\ +2\sin\alpha\displaystyle\sum_{j=1,\,j\ne\lambda}^{n-1}|a_j|\}\end{array}\right]\quad\text{for } R\ge1$$

and

$$\frac{1}{\log c}\log\frac{1}{|a_0|}\left[\begin{array}{l}|a_n|R^{n+1}+|a_0|+R^n\{|a_n|(\cos\alpha+\sin\alpha)+2|a_\lambda|(k+k\sin\alpha-1)\\ -\tau|a_0|(\cos\alpha-\sin\alpha+1)+|a_0|\ +2\sin\alpha\displaystyle\sum_{j=1,\,j\ne\lambda}^{n-1}|a_j|\}\end{array}\right]\quad\text{for } R\le1,$$

where

$$M_2=|a_n|(\cos\alpha+\sin\alpha+1)+2|a_\lambda|(k+k\sin\alpha-1)-\tau|a_0|(\cos\alpha-\sin\alpha+1)+|a_0|$$
$$+2\sin\alpha\sum_{j=1,\,j\ne\lambda}^{n-1}|a_j|.$$

Theorem 3: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$,

$\mathrm{Im}(a_j)=\beta_j,\ j=0,1,......,n$ such that for some $k_1\ge1, k_2\ge1, 0<\tau\le1$,

$$k_1\alpha_n\ge k_2\alpha_{n-1}\ge\alpha_{n-2}......\alpha_1\ge\tau\alpha_0$$

Then the number of zeros of P(z) in $|z|\le\frac{R}{c}\,(R>0,c>1)$ does not exceed

$$\frac{1}{\log c}\log\frac{1}{|a_0|}\left[\begin{array}{l}|a_n|R^{n+1}+R^n\{(k_1|\alpha_n|+k_2|\alpha_{n-1}|)+(k_1\alpha_n-k_2\alpha_{n-1})+\alpha_{n-1}-(|\alpha_n|+|\alpha_{n-1}|)\\ -\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+|\beta_n|+2\sum_{j=1}^{n-1}|\beta_j|\}\end{array}\right]$$

for $R\geq 1$ and

$$\frac{1}{\log c}\log\frac{1}{|a_0|}\left[\begin{array}{l}|a_n|R^{n+1}+R\{(k_1|\alpha_n|+k_2|\alpha_{n-1}|)+(k_1\alpha_n-k_2\alpha_{n-1})+\alpha_{n-1}-(|\alpha_n|+|\alpha_{n-1}|)\\ -\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+|\beta_n|+2\sum_{j=1}^{n-1}|\beta_j|\}\end{array}\right]$$

for $R\leq 1$.

Combining Theorem 3 and Theorem F, we get a bound for the number of zeros of P(z) in a ring-shaped region as follows:

Corollary 3: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j,\ j=0,1,......,n$ such that for some $k_1\geq 1, k_2\geq 1, 0<\tau\leq 1$,

$$k_1\alpha_n\geq k_2\alpha_{n-1}\geq\alpha_{n-2}......\alpha_1\geq\tau\alpha_0$$

Then the number of zeros of P(z) in $\frac{|a_0|}{M_3}\leq|z|\leq\frac{R}{c}\ (R>0, c>1)$ does not exceed

$$\frac{1}{\log c}\log\frac{1}{|a_0|}\left[\begin{array}{l}|a_n|R^{n+1}+R^n\{(k_1|\alpha_n|+k_2|\alpha_{n-1}|)+(k_1\alpha_n-k_2\alpha_{n-1})+\alpha_{n-1}-(|\alpha_n|+|\alpha_{n-1}|)\\ -\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+|\beta_n|+2\sum_{j=1}^{n-1}|\beta_j|\}\end{array}\right]$$

for $R\geq 1$ and

$$\frac{1}{\log c}\log\frac{1}{|a_0|}\left[\begin{array}{l}|a_n|R^{n+1}+R\{(k_1|\alpha_n|+k_2|\alpha_{n-1}|)+(k_1\alpha_n-k_2\alpha_{n-1})+\alpha_{n-1}-(|\alpha_n|+|\alpha_{n-1}|)\\ -\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+|\beta_n|+2\sum_{j=1}^{n-1}|\beta_j|\}\end{array}\right]$$

for $R\leq 1$, where

$$\begin{aligned}M_3=|a_n|&+(k_1|\alpha_n|+k_2|\alpha_{n-1}|)+(k_1\alpha_n-k_2\alpha_{n-1})+\alpha_{n-1}-(|\alpha_n|+|\alpha_{n-1}|)\\ &-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+|\beta_n|+2\sum_{j=1}^{n-1}|\beta_j|.\end{aligned}$$

Theorem 4: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n such that for some $k_1\geq 1, k_2\geq 1, 0<\tau\leq 1$,

$$k_1|a_n|\geq k_2|a_{n-1}|\geq|a_{n-2}|......\geq|a_1|\geq\tau|a_0|.$$

Then the number of zeros of P(z) in $|z|\leq\frac{R}{c}\ (R>0, c>1)$ does not exceed

$$\frac{1}{\log c}\log\frac{1}{|a_0|}\left[\begin{array}{l}|a_n|R^{n+1}+|a_0|+R^n\{k_1|a_n|(\cos\alpha+\sin\alpha+1)+k_2|a_{n-1}|(\sin\alpha-\cos\alpha+1)-|a_n|\\ -|a_{n-1}|(1-\cos\alpha-\sin\alpha)-\tau|a_0|(\cos\alpha-\sin\alpha+1)+2\sin\alpha\sum_{j=1}^{n-2}|a_j|\}\end{array}\right]$$

for $R\geq 1$ and

$$\frac{1}{\log c}\log\frac{1}{|a_0|}\left[\begin{array}{l}|a_n|R^{n+1}+|a_0|+R\{k_1|a_n|(\cos\alpha+\sin\alpha+1)+k_2|a_{n-1}|(\sin\alpha-\cos\alpha+1)-|a_n|\\ -|a_{n-1}|(1-\cos\alpha-\sin\alpha)-\tau|a_0|(\cos\alpha-\sin\alpha+1)+2\sin\alpha\sum_{j=1}^{n-2}|a_j|\}\end{array}\right]$$

for $R\le 1$.

Combining Theorem 4 and Theorem G, we get a bound for the number of zeros of P(z) in a ring-shaped region as follows:

Corollary 4: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n such that for some $k_1\ge 1, k_2\ge 1, 0<\tau\le 1$,

$$k_1|a_n|\ge k_2|a_{n-1}|\ge|a_{n-2}|......\ge|a_1|\ge\tau|a_0|.$$

Then the number of zeros of P(z) in $\frac{|a_0|}{M_4}\le|z|\le\frac{R}{c}\,(R>0, c>1)$ does not exceed

$$\frac{1}{\log c}\log\frac{1}{|a_0|}\left[\begin{array}{l}|a_n|R^{n+1}+|a_0|+R^n\{k_1|a_n|(\cos\alpha+\sin\alpha+1)+k_2|a_{n-1}|(\sin\alpha-\cos\alpha+1)-|a_n|\\ -|a_{n-1}|(1-\cos\alpha-\sin\alpha)-\tau|a_0|(\cos\alpha-\sin\alpha+1)+2\sin\alpha\sum_{j=1}^{n-2}|a_j|\}\end{array}\right]$$

for $R\ge 1$ and

$$\frac{1}{\log c}\log\frac{1}{|a_0|}\left[\begin{array}{l}|a_n|R^{n+1}+|a_0|+R\{k_1|a_n|(\cos\alpha+\sin\alpha+1)+k_2|a_{n-1}|(\sin\alpha-\cos\alpha+1)-|a_n|\\ -|a_{n-1}|(1-\cos\alpha-\sin\alpha)-\tau|a_0|(\cos\alpha-\sin\alpha+1)+2\sin\alpha\sum_{j=1}^{n-2}|a_j|\}\end{array}\right]$$

for $R\le 1$, where

$$M_4=k_1|a_n|(\cos\alpha+\sin\alpha+1)+k_2|a_{n-1}|(\sin\alpha-\cos\alpha+1)-|a_{n-1}|(1-\cos\alpha)$$
$$-\tau|a_0|(\cos\alpha-\sin\alpha+1)+|a_0|.$$

For different values of the parameters in the above results, we get many interesting results including generalizations of some well-known results.

II. LEMMAS

For the proofs of the above results, we need the following results:

Lemma 1: Let $z_1, z_2\in C$ with $|z_1|\ge|z_2|$ and $|\arg z_j-\beta|\le\alpha\le\frac{\pi}{2}, j=1,2$ for some real numbers α and β. Then

$$|z_1-z_2|\le(|z_1|-|z_2|)\cos\alpha+(|z_1|+|z_2|)\sin\alpha.$$

The above lemma is due to Govil and Rahman [2].

Lemma 2: Let F(z) be analytic in $|z|\le R$, $|F(z)|\le M$ for $|z|\le R$ and $F(0)\ne 0$. Then for c>1, the number of zeros of F(z) in the disk $|z|\le\frac{R}{c}$ does not exceed

$$\frac{1}{\log c}\log\frac{M}{|a_0|}.$$

For the proof of this lemma see [7].

III. PROOFS OF THEOREMS

Proof of Theorem 1: Consider the polynomial

F(z)=(1-z)P(z)

$$=(1-z)(a_nz^n+a_{n-1}z^{n-1}+......+a_{\lambda+1}z^{\lambda+1}+a_\lambda z^\lambda+a_{\lambda-1}z^{\lambda-1}+......+a_{n-1}z^{n-1}+a_nz^n$$

$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + ... + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda$$
$$+ (a_{\lambda-1} - a_{\lambda-2})z^{\lambda-1} + ... + (a_1 - a_0)z + a_0$$
$$= -a_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + ...$$
$$+ ...... + \{(\alpha_{\lambda+1} - k\alpha_\lambda) + (k\alpha_\lambda - \alpha_\lambda)\}z^{\lambda+1} + \{(k\alpha_\lambda - \alpha_{\lambda-1}) -$$
$$(k\alpha_\lambda - \alpha_\lambda)\}z^\lambda + (\alpha_{\lambda-1} - \alpha_{\lambda-2})z^{\lambda-1} + ... + \{(\alpha_1 - \tau\alpha_0) + (\tau\alpha_0 - \alpha_0)\}z$$
$$+ i\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j + a_0$$

For $|z| \le R$, we have, by using the hypothesis,

$$|F(z)| \le |a_n|R^{n+1} + |\alpha_n - \alpha_{n-1}|R^n + |\alpha_{n-1} - \alpha_{n-2}|R^{n-1} + ... + |\alpha_{\lambda+1} - k\alpha_\lambda|R^{\lambda+1} + (k-1)|\alpha_\lambda|R^{\lambda+1}$$
$$+ |k\alpha_\lambda - \alpha_{\lambda-1}|R^\lambda + (k-1)|\alpha_\lambda|R^\lambda + |\alpha_{\lambda-1} - \alpha_{\lambda-2}|R^{\lambda-1} + ... + |\alpha_1 - \tau\alpha_0|R$$
$$+ (1-\tau)|\alpha_0|R + \sum_{j=1}^{n}(|\beta_j| + |\beta_{j-1}|).R^j + |a_0|$$
$$\le |a_n|R^{n+1} + |a_0| + R^n[\alpha_n - \alpha_{n-1} + \alpha_{n-1} - \alpha_{n-2} + ... + \alpha_{\lambda+1} - k\alpha_\lambda + (k-1)|\alpha_\lambda|$$
$$+ k\alpha_\lambda - \alpha_{\lambda-1} + (k-1)|\alpha_\lambda| + \alpha_{\lambda-1} - \alpha_{\lambda-2} + ... + \alpha_1 - \tau\alpha_0$$
$$+ (1-\tau)|\alpha_0| + \sum_{j=1}^{n}(|\beta_j| + |\beta_{j-1}|)]$$
$$= |a_n|R^{n+1} + |a_0| + R^n[\alpha_n + 2(k-1)|\alpha_\lambda| - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n}|\beta_j|]$$

for $R \ge 1$

and

$$|F(z)| \le |a_n|R^{n+1} + |a_0| + R[\alpha_n - \alpha_{n-1} + \alpha_{n-1} - \alpha_{n-2} + ... + \alpha_{\lambda+1} - k\alpha_\lambda + (k-1)|\alpha_\lambda|$$
$$+ k\alpha_\lambda - \alpha_{\lambda-1} + (k-1)|\alpha_\lambda| + \alpha_{\lambda-1} - \alpha_{\lambda-2} + ... + \alpha_1 - \tau\alpha_0$$
$$+ (1-\tau)|\alpha_0| + \sum_{j=1}^{n}(|\beta_j| + |\beta_{j-1}|)]$$
$$= |a_n|R^{n+1} + |a_0| + R[\alpha_n + 2(k-1)|\alpha_\lambda| - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n}|\beta_j|]$$

for $R \le 1$.

Hence, by Lemma 2, it follows that the number of zeros of F(z) in $|z| \le \dfrac{R}{c}, c > 1$ does not exceed

$$\frac{1}{\log c}\log\frac{|a_n|R^{n+1} + |a_0| + R^n[\alpha_n + 2(k-1)|\alpha_\lambda| - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n}|\beta_j|]}{|a_0|}$$

for $R \ge 1$ and

$$\frac{1}{\log c}\log\frac{|a_n|R^{n+1} + |a_0| + R[\alpha_n + 2(k-1)|\alpha_\lambda| - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n}|\beta_j|]}{|a_0|}$$

for $R \le 1$.
Since the zeros of P(z) are also the zeros of F(z), the proof of Theorem 1is complete.
Proof of Theorem 2: Consider the polynomial
F(z)=(1-z)P(z)

$$= (1-z)(a_n z^n + a_{n-1}z^{n-1} + ...+ a_{\lambda+1}z^{\lambda+1} + a_\lambda z^\lambda + a_{\lambda-1}z^{\lambda-1} + ...+ a_{n-1}z^{n-1} + a_n z^n$$
$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + ...+ (a_{\lambda+1} - a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda$$
$$+ (a_{\lambda-1} - a_{\lambda-2})z^{\lambda-1} + ...+ (a_1 - a_0)z + a_0$$

For $|z| \le R$, we have, by using the hypothesis and Lemma 1,

$$\begin{aligned}
|F(z)| &\le |a_n|R^{n+1} + |a_n - a_{n-1}|R^n + |a_{n-1} - a_{n-2}|R^{n-1} + ...+ |a_{\lambda+1} - ka_\lambda|R^{\lambda+1} + (k-1)|a_\lambda|R^{\lambda+1} \\
&\quad + |ka_\lambda - a_{\lambda-1}|R^\lambda + (k-1)|a_\lambda|R^\lambda + |a_{\lambda-1} - a_{\lambda-2}|R^{\lambda-1} + ...+ |a_1 - \tau a_0|R \\
&\quad + (1-\tau)|a_0|R \\
&\le |a_n|R^{n+1} + |a_0| + R^n[(|a_n| - |a_{n-1}|)\cos\alpha + (|a_n| + |a_{n-1}|)\sin\alpha + (|a_{n-1}| - |a_{n-2}|)\cos\alpha \\
&\quad + (|a_{n-1}| + |a_{n-2}|)\sin\alpha + ...+ (|a_{\lambda+1}| - k|a_\lambda|)\cos\alpha + (|a_{\lambda+1}| + k|a_\lambda|)\sin\alpha \\
&\quad + (k-1)|\alpha_\lambda| + (k|\alpha_\lambda| - |\alpha_{\lambda-1}|)\cos\alpha + (k|\alpha_\lambda| + |\alpha_{\lambda-1}|)\sin\alpha + (k-1)|\alpha_\lambda| \\
&\quad + (|a_{\lambda-1}| - |a_{\lambda-2}|)\cos\alpha + (|a_{\lambda-1}| + |a_{\lambda-2}|)\sin\alpha + ...+ (|a_1| - \tau|a_0|)\cos\alpha \\
&\quad + (|a_1| + \tau|a_0|)\sin\alpha + (1-\tau)|a_0|] \\
&= |a_n|R^{n+1} + |a_0| + R^n[|a_n|(\cos\alpha + \sin\alpha) + 2|a_\lambda|(k + k\sin\alpha - 1) \\
&\quad - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1, j\ne\lambda}^{n-1}|a_j|]
\end{aligned}$$

for $R \ge 1$

and

$$\begin{aligned}
|F(z) \le& |a_n|R^{n+1} + |a_0| + R[|a_n|(\cos\alpha + \sin\alpha) + 2|a_\lambda|(k + k\sin\alpha - 1) \\
&- \tau|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1, j\ne\lambda}^{n-1}|a_j|]
\end{aligned}$$

for $R \le 1$.

Hence, by Lemma 2, it follows that the number of zeros of F(z) in $|z| \le \frac{R}{c}, c > 1$
does not exceed

$$\frac{1}{\log c}\log\frac{1}{|a_0|}\begin{bmatrix} |a_n|R^{n+1} + |a_0| + R^n\{|a_n|(\cos\alpha + \sin\alpha) + 2|a_\lambda|(k + k\sin\alpha - 1) \\ -\tau|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1, j\ne\lambda}^{n-1}|a_j|\} \end{bmatrix} \quad \text{for } R \ge 1$$

and

$$\frac{1}{\log c}\log\frac{1}{|a_0|}\begin{bmatrix} |a_n|R^{n+1} + |a_0| + R^n\{|a_n|(\cos\alpha + \sin\alpha) + 2|a_\lambda|(k + k\sin\alpha - 1) \\ -\tau|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1, j\ne\lambda}^{n-1}|a_j|\} \end{bmatrix} \text{for } R \le 1.$$

Since the zeros of P(z) are also the zeros of F(z), Theorem 2 follows.
Proof of Theorem 3: Consider the polynomial

$$F(z) = (1-z)P(z) = (1-z)(a_n z^n + a_{n-1}z^{n-1} + ...+ a_1 z + a_0)$$
$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + ...+ (a_1 - a_0)z + a_0$$

$$= -a_n z^{n+1} + \{(k_1\alpha_n - k_2\alpha_{n-1}) - (k_1\alpha_n - \alpha_n) + (k_2\alpha_{n-1} - \alpha_{n-1})\}z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1}$$
$$+ ...... + \{(\alpha_1 - \tau\alpha_0) + (\tau\alpha_0 - \alpha_0)\}z + i\{(\beta_n - \beta_{n-1})z^n + ...$$
$$+ (\beta_1 - \beta_0)z\} + a_0$$

For $|z| \le R$, we have, by using the hypothesis,

$$|F(z)| \le |a_n|R^{n+1} + |k_1\alpha_n - k_2\alpha_{n-1}|R^n + (k_1 - 1)|\alpha_n|R^n + (k_2 - 1)|\alpha_{n-1}|R^n + |\alpha_{n-1} - \alpha_{n-2}|R^{n-1} + ...$$
$$+ |\alpha_1 - \tau\alpha_0|R + (1-\tau)|\alpha_0|R + \sum_{j=1}^{n}|\beta_j - \beta_{j-1}|R^j + |a_0|$$
$$\le |a_n|R^{n+1} + R^n[k_1\alpha_n - k_2\alpha_{n-1} + (k_1 - 1)|\alpha_n| + (k_2 - 1)|\alpha_{n-1}| + \alpha_{n-1} - \alpha_{n-2} + ...$$
$$+ \alpha_1 - \tau\alpha_0 + (1-\tau)|\alpha_0| + \sum_{j=1}^{n}(|\beta_j| + |\beta_{j-1}|)]$$
$$= |a_n|R^{n+1} + R^n[(k_1|\alpha_n| + k_2|\alpha_{n-1}|) + (k_1\alpha_n - k_2\alpha_{n-1}) + \alpha_{n-1} - (|\alpha_n| + |\alpha_{n-1}|)$$
$$- \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n-1}|\beta_j|]$$

for $R \ge 1$

and

$$|F(z)| \le |a_n|R^{n+1} + R[(k_1|\alpha_n| + k_2|\alpha_{n-1}|) + (k_1\alpha_n - k_2\alpha_{n-1}) + \alpha_{n-1} - (|\alpha_n| + |\alpha_{n-1}|)$$
$$- \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n-1}|\beta_j|]$$

for $R \le 1$.

Hence, by Lemma 2, it follows that the number of zeros of F(z) in $|z| \le \dfrac{R}{c}, c > 1$ does not exceed

$$\frac{1}{\log c}\log\frac{1}{|a_0|}\left[\begin{matrix}|a_n|R^{n+1} + R^n\{(k_1|\alpha_n| + k_2|\alpha_{n-1}|) + (k_1\alpha_n - k_2\alpha_{n-1}) + \alpha_{n-1} - (|\alpha_n| + |\alpha_{n-1}|) \\ - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n-1}|\beta_j|\}\end{matrix}\right]$$

for $R \ge 1$ and

$$\frac{1}{\log c}\log\frac{1}{|a_0|}\left[\begin{matrix}|a_n|R^{n+1} + R\{(k_1|\alpha_n| + k_2|\alpha_{n-1}|) + (k_1\alpha_n - k_2\alpha_{n-1}) + \alpha_{n-1} - (|\alpha_n| + |\alpha_{n-1}|) \\ - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n-1}|\beta_j|\}\end{matrix}\right]$$

for $R \le 1$.

Since the zeros of P(z) are also the zeros of F(z), Theorem 3 follows.

Proof of Theorem 4: Consider the polynomial

$$F(z) = (1-z)P(z) = (1-z)(a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0)$$
$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + ... + (a_1 - a_0)z + a_0$$
$$= -a_n z^{n+1} + \{(k_1 a_n - k_2 a_{n-1}) - (k_1 a_n - a_n) + (k_2 a_{n-1} - a_{n-1})\}z^n + (a_{n-1} - a_{n-2})z^{n-1}$$
$$+ ...... + \{(a_1 - \tau a_0) + (\tau a_0 - a_0)\}z + a_0$$

For $|z| \le R$, we have, by using the hypothesis and Lemma 1,

$$|F(z)| \le |a_n|R^{n+1} + |k_1 a_n - k_2 a_{n-1}|R^n + (k_1 - 1)|a_n|R^n + (k_2 - 1)|a_{n-1}|R^n + |a_{n-1} - a_{n-2}|R^{n-1} + \ldots\ldots$$

$$+ |a_1 - \tau a_0|R + (1-\tau)|a_0|R + |a_0|$$

$$\le |a_n|R^{n+1} + |a_0| + R^n[(k_1|a_n| - k_2|a_{n-1}|)\cos\alpha + (k_1|a_n| + k_2|a_{n-1}|)\sin\alpha + (k_1 - 1)|a_n|$$

$$+ (k_2 - 1)|a_{n-1}| + (|a_{n-1}| - |a_{n-2}|)\cos\alpha + (|a_{n-1}| + |a_{n-2}|)\sin\alpha$$

$$+ (|a_1| - \tau|a_0|)\cos\alpha + (|a_1| + \tau|a_0|)\sin\alpha + (1-\tau)|a_0|$$

$$= |a_n|R^{n+1} + |a_0| + R^n[k_1|a_n|(\cos\alpha + \sin\alpha + 1) + k_2|a_{n-1}|(\sin\alpha - \cos\alpha + 1) - |a_n|$$

$$- |a_{n-1}|(1 - \cos\alpha - \sin\alpha) - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2\sin\alpha\sum_{j=1}^{n-2}|a_j|]$$

for $R \ge 1$

and

$$|F(z)| \le |a_n|R^{n+1} + |a_0| + R[k_1|a_n|(\cos\alpha + \sin\alpha + 1) + k_2|a_{n-1}|(\sin\alpha - \cos\alpha + 1) - |a_n|$$

$$- |a_{n-1}|(1 - \cos\alpha - \sin\alpha) - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2\sin\alpha\sum_{j=1}^{n-2}|a_j|]$$

for $R \le 1$.

Hence, by Lemma 2, it follows that the number of zeros of F(z) in $|z| \le \frac{R}{c}, c > 1$

does not exceed

$$\frac{1}{\log c}\log\frac{1}{|a_0|}\left[\begin{array}{l}|a_n|R^{n+1} + |a_0| + R^n\{k_1|a_n|(\cos\alpha + \sin\alpha + 1) + k_2|a_{n-1}|(\sin\alpha - \cos\alpha + 1) - |a_n| \\ - |a_{n-1}|(1 - \cos\alpha - \sin\alpha) - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2\sin\alpha\sum_{j=1}^{n-2}|a_j|\}\end{array}\right]$$

for $R \ge 1$ and

$$\frac{1}{\log c}\log\frac{1}{|a_0|}\left[\begin{array}{l}|a_n|R^{n+1} + |a_0| + R\{k_1|a_n|(\cos\alpha + \sin\alpha + 1) + k_2|a_{n-1}|(\sin\alpha - \cos\alpha + 1) - |a_n| \\ - |a_{n-1}|(1 - \cos\alpha - \sin\alpha) - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2\sin\alpha\sum_{j=1}^{n-2}|a_j|\}\end{array}\right]$$

for $R \le 1$.

Since the zeros of P(z) are also the zeros of F(z), Theorem 4 follows.

REFERENCES

[1]. K.K. Dewan, Extremal Properties and Coefficient Estimates for Polynomials with Restricted Coefficients and on Location of Zeros of Polynomials,Ph. Thesis, IIT Delhi, 1980.

[2]. N. K. Govil and Q. I. Rahman, On the Enestrom-Kakeya Theorem, Tohoku Mathematics Journal, 20(1968), 136.

[3]. M. H. Gulzar, The Number of Zeros of a Polynomial in a Disk,International Journal of Innovative Science, Engineering & Technology, Vol.1, Issue 2,April 2014, 37-46.

[4]. M. H. Gulzar, Zeros of Polynomials under Certain Coefficient Conditions, Int. Journal of Advance Foundation and Research in Science and Technology (to appear) .

[5] M. Marden, , Geometry of Polynomials,Math. Surveys No. 3, Amer. Math Society(R.I. Providence) (1966).

[6] Q. G. Mohammad, On the Zeros of Polynomials, American Mathematical Monthly, 72 (6) (1965), 631-633.

[7] E. C. Titchmarsh, The Theory of Functions, 2nd Edition, Oxford University Press,London, 1939.