International Journal of Engineering Research and Development
e-ISSN: 2278-067X, p-ISSN: 2278-800X, www.ijerd.com
Volume 10, Issue 5 (May 2014), PP.75-81

On Some Extensions of Enestrom-Kakeya Theorem

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Abstract: In this paper we generalize some recent extensions of the Enestrom-Kakeya Theorem concerning the location of zeros of polynomials.

Mathematics Subject Classification: 30 C 10, 30 C 15

Keywords and Phrases: Coefficient, Polynomial, Zero

I. INTRODUCTION AND STATEMENT OF RESULTS

The following result known as the Enestrom-Kakeya Theorem [7] is an elegant result in the theory of distribution of the zeros of a polynomial:

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0 > 0.$$

Then all the zeros of P(z) lie in the closed disk $|z| \leq 1$.

In the literature [1-8], there exist several extensions and generalizations of Theorem A. Joyal, Labelle and Rahman [5] gave the following generalization of Theorem A:

Theorem B: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0 .$$

Then all the zeros of P(z) lie in the closed disk $|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}$.

Aziz and Zargar [1] generalized Theorems A and B by proving the following results:

Theorem C: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k \geq 1$,

$$ka_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0 .$$

Then all the zeros of P(z) lie in the closed disk

$$|z+k-1| \leq \frac{ka_n - a_0 + |a_0|}{|a_n|} .$$

Recently, Zargar [8] gave the following generalizations of the above mentioned results:

Theorem D: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n≥ 2 such that for some $k \geq 1$, either

$ka_n \geq a_{n-2} \geq .... \geq a_3 \geq a_1 > 0$ and $a_{n-1} \geq a_{n-3} \geq .... \geq a_2 \geq a_0 > 0$,
if n is odd or
$ka_n \geq a_{n-2} \geq .... \geq a_2 \geq a_0 > 0$ and $a_{n-1} \geq a_{n-3} \geq .... \geq a_3 \geq a_1 > 0$,
if n is even.

Then all the zeros of P(z) lie in the region

$$|z-\alpha||z-\beta| \leq k + \frac{a_{n-1}}{a_n} ,$$

where α, β are the roots of the quadratic

$$z^2 + \frac{a_{n-1}}{a_n} z + k - 1 = 0.$$

Theorem E: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n≥ 2 such that

either

$$a_n \geq a_{n-2} \geq .... \geq a_{2\lambda+1} \leq a_{2\lambda-1} \leq .... \leq a_3 \leq a_1 > 0 \text{ and}$$
$$a_{n-1} \geq a_{n-3} \geq .... \geq a_{2\lambda} \leq a_{2\lambda-2} \leq ...... \leq a_2 \leq a_0 > 0,$$

for some integer $\lambda, 0 \leq \lambda \leq \frac{n-1}{2}$, if n is odd

or

$$a_n \geq a_{n-2} \geq .... \geq a_{2\lambda} \leq a_{2\lambda-2} \leq ...... a_2 \leq a_0 > 0 \text{ and}$$
$$a_{n-1} \geq a_{n-3} \geq .... \geq a_{2\lambda+1} \leq a_{2\lambda-1} \leq a_3 \leq a_1 > 0,$$

for some integer $\lambda, 0 \leq \lambda \leq \frac{n-2}{2}$, if n is even.

Then all the zeros of P(z) lie in the closed disk

$$\left| z + \frac{a_{n-1}}{a_n} \right| \leq 1 + \frac{a_{n-1} + 2(a_0 + a_1 - a_{2\lambda} - 2a_{2\lambda+1})}{a_n}$$

In this paper, we give generalizations of Theorems D and E and prove the following results:

Theorem 1: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n≥ 2 such that for some $k_1, k_2 \geq 1; 0 < \tau_1, \tau_2 \leq 1$, either

$k_1 a_n \geq a_{n-2} \geq .... \geq a_3 \geq \tau_1 a_1 > 0$ and $k_2 a_{n-1} \geq a_{n-3} \geq .... \geq a_2 \geq \tau_2 a_0 > 0$,

if n is odd or

$k_1 a_n \geq a_{n-2} \geq .... \geq a_2 \geq \tau_1 a_0 > 0$ and $k_2 a_{n-1} \geq a_{n-3} \geq .... \geq a_3 \geq \tau_2 a_1 > 0$,

if n is even.

Then for odd n all the zeros of P(z) lie in the region

$$|z - \alpha||z - \beta| \leq \frac{k_1 a_n + (2k_2 - 1)a_{n-1} - 2(\tau_1 - 1)a_1 - 2(\tau_2 - 1)a_0}{a_n}$$

and for even n all the zeros of P(z) lie in the region

$$|z - \alpha||z - \beta| \leq \frac{k_1 a_n + (2k_2 - 1)a_{n-1} - 2(\tau_1 - 1)a_0 - 2(\tau_2 - 1)a_1}{a_n},$$

where α, β are the roots of the quadratic

$$z^2 + \frac{a_{n-1}}{a_n} z + k_1 - 1 = 0.$$

Remark 1: For $k_1 = k, k_2 = 1, \tau_1 = 1, \tau_2 = 1$, Theorem 1 reduces to Theorem D of Zargar.

Taking $a_{n-1} = 2a_n\sqrt{k_1 - 1}$, $k_2 = 1$ and noting that the quadratic $z^2 + 2\sqrt{k_1 - 1}z + k_1 - 1 = 0$ has two equal roots each equal to $-\sqrt{k_1 - 1}$,we get the following

Corollary 1: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n≥ 2 such that for some $k_1 \geq 1; 0 < \tau_1, \tau_2 \leq 1$, either

$$k_1 a_n \geq a_{n-2} \geq .... \geq a_3 \geq \tau_1 a_1 > 0 \text{ and}$$

$$2a_n\sqrt{k_1-1}=a_{n-1}\ge a_{n-3}\ge......\ge a_2\ge\tau_2a_0>0,$$

if n is odd,
or

$$k_1a_n\ge a_{n-2}\ge.......\ge a_2\ge\tau_1a_0>0 \text{ and}$$

$$2a_n\sqrt{k_1-1}=a_{n-1}\ge a_{n-3}\ge......\ge a_3\ge\tau_2a_1>0,$$

if n is even.
Then for odd n all the zeros of P(z) lie in the region

$$\left|z+\sqrt{k_1-1}\right|\le\left\{\frac{k_1a_n+2a_n\sqrt{k_1-1}-2(\tau_1-1)a_1-2(\tau_2-1)a_0}{a_n}\right\}^{\frac{1}{2}}$$

and for even n all the zeros of P(z) lie in the region

$$\left|z+\sqrt{k_1-1}\right|\le\left\{\frac{k_1a_n+2a_n\sqrt{k_1-1}-2(\tau_1-1)a_0-2(\tau_2-1)a_1}{a_n}\right\}^{\frac{1}{2}}.$$

For $k_1=k,\tau_1=1,\tau_2=1$, Cor. 1 reduces to a result of Zargar [8, Cor.1] .
Applying Cor. 1 to a polynomial of even degree, we get the following

Corollary 2: Let $Q(z)=\sum_{j=0}^{2n}b_jz^j$ be a polynomial of even degree 2n such that for some $k_1\ge1;0<\tau_1,\tau_2\le1$,

$$k_1b_{2n}\ge b_{2n-2}\ge.......\ge b_2\ge\tau_1b_0>0 \text{ and}$$

$$2\sqrt{k_1-1}b_{2n}=b_{2n-1}\ge b_{2n-3}\ge......\ge b_3\ge\tau_2b_1>0.$$

Then all the zeros of Q(z) lie in

$$\left|z+\sqrt{k_1-1}\right|\le\left\{\frac{k_1b_{2n}+2b_{2n}\sqrt{k_1-1}-2(\tau_1-1)b_0-2(\tau_2-1)b_1}{b_{2n}}\right\}^{\frac{1}{2}}.$$

Taking $k_1=1,\tau_1=1,\tau_2=1$, Cor. 2 reduces to Enestrom-Kakeya Theorem (see [8]).

Taking $k_1=2,$ we get the following result from Cor.1:

Corollary 3: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n≥2 such that for some $0<\tau_1,\tau_2\le1$, either

$$2a_n\ge a_{n-2}\ge.......\ge a_3\ge\tau_1a_1>0 \text{ and}$$

$$2a_n=a_{n-1}\ge a_{n-3}\ge.......\ge a_2\ge\tau_2a_0>0,$$

if n is odd,
or

$$2a_n\ge a_{n-2}\ge.......\ge a_2\ge\tau_1a_0>0 \text{ and}$$

$$2a_n=a_{n-1}\ge a_{n-3}\ge.......\ge a_3\ge\tau_2a_1>0,$$

if n is even.
Then for odd n all the zeros of P(z) lie in the region

$$|z+1|\le\left\{\frac{2-2(\tau_1-1)a_1-2(\tau_2-1)a_0}{a_n}\right\}^{\frac{1}{2}}$$

and for even n all the zeros of P(z) lie in the region

$$|z+1|\le\left\{\frac{2-2(\tau_1-1)a_0-2(\tau_2-1)a_1}{a_n}\right\}^{\frac{1}{2}}.$$

For $\tau_1 = 1, \tau_2 = 1$, Cor.3 reduces to a result of Zargar [8,Cor.3].

Theorem 2: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n≥ 2 such that for some $k_1, k_2 \geq 1; 0 < \tau_1, \tau_2 \geq 1$, either

$$k_1 a_n \geq a_{n-2} \geq .... \geq a_{2\lambda+1} \leq a_{2\lambda-1} \leq .... \leq a_3 \leq \tau_1 a_1 > 0 \text{ and}$$

$$k_2 a_{n-1} \geq a_{n-3} \geq .... \geq a_{2\lambda} \leq a_{2\lambda-2} \leq ...... \leq a_2 \leq \tau_2 a_0 > 0,$$

for some integer $\lambda, 0 \leq \lambda \leq \frac{n-1}{2}$, if n is odd

or

$$k_1 a_n \geq a_{n-2} \geq .... \geq a_{2\lambda} \leq a_{2\lambda-2} \leq ...... a_2 \leq \tau_1 a_0 > 0 \text{ and}$$

$$k_2 a_{n-1} \geq a_{n-3} \geq .... \geq a_{2\lambda+1} \leq a_{2\lambda-1} \leq a_3 \leq \tau_2 a_1 > 0,$$

for some integer $\lambda, 0 \leq \lambda \leq \frac{n-2}{2}$, if n is even.

Then for odd n all the zeros of P(z) lie in the disk

$$\left| z + \frac{a_{n-1}}{a_n} \right| \leq \frac{(2k_1 - 1)a_n + (2k_2 - 1)a_{n-1} - 2a_{2\lambda} - 2a_{2\lambda+1} + 2\tau_1 a_1 + 2\tau_2 a_0}{a_n}$$

and for even n all the zeros of P(z) lie in the disk

$$\left| z + \frac{a_{n-1}}{a_n} \right| \leq \frac{(2k_1 - 1)a_n + (2k_2 - 1)a_{n-1} - 2a_{2\lambda} - 2a_{2\lambda+1} + 2\tau_1 a_0 + 2\tau_2 a_1}{a_n}.$$

Remark 2: For $k_1 = 1, k_2 = 1, \tau_1 = 1, \tau_2 = 1$, Theorem 2 reduces to Theorem E of Zargar.

Applying Theorem 2 to the polynomial P(tz), we get the following result:

Corollary 4: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n≥ 2 such that for some $k_1, k_2 \geq 1; 0 < \tau_1, \tau_2 \geq 1$ and t>0, either

$$k_1 a_n t^n \geq a_{n-2} t^{n-2} \geq .... \geq a_{2\lambda+1} t^{2\lambda+1} \leq a_{2\lambda-1} t^{2\lambda-1} \leq .... \leq a_3 t^3 \leq \tau_1 a_1 t > 0 \text{ and}$$

$$k_2 a_{n-1} t^{n-1} \geq a_{n-3} t^{n-3} \geq .... \geq a_{2\lambda} t^{2\lambda} \leq a_{2\lambda-2} t^{2\lambda-2} \leq ...... \leq a_2 t^2 \leq \tau_2 a_0 > 0,$$

for some integer $\lambda, 0 \leq \lambda \leq \frac{n-1}{2}$, if n is odd

or

$$k_1 a_n t^n \geq a_{n-2} t^{n-2} \geq .... \geq a_{2\lambda} t^{2\lambda} \leq a_{2\lambda-2} t^{2\lambda-2} \leq ...... \leq a_2 t^2 \leq \tau_1 a_0 > 0 \text{ and}$$

$$k_2 a_{n-1} t^{n-1} \geq a_{n-3} t^{n-3} \geq .... \geq a_{2\lambda+1} t^{2\lambda+1} \leq a_{2\lambda-1} t^{2\lambda-1} \leq ...... \leq a_3 t^3 \leq \tau_2 a_1 t > 0,$$

for some integer $\lambda, 0 \leq \lambda \leq \frac{n-2}{2}$, if n is even.

Then for odd n all the zeros of P(z) lie in the disk

$$\left| z + \frac{a_{n-1}}{a_n} \right| \leq \frac{(2k_1 - 1)a_n t^{n-1} + (2k_2 - 1)a_{n-1} t^{n-1} - 2a_{2\lambda} t^{2\lambda} - 2a_{2\lambda+1} t^{2\lambda} + 2\tau_1 a_1 t + 2\tau_2 a_0}{t^{n-1} a_n}$$

and for even n all the zeros of P(z) lie in the disk

$$\left| z + \frac{a_{n-1}}{a_n} \right| \leq \frac{(2k_1 - 1)a_n t^{n-1} + (2k_2 - 1)a_{n-1} t^{n-1} - 2a_{2\lambda} t^{2\lambda} - 2a_{2\lambda+1} t^{2\lambda} + 2\tau_1 a_0 + 2\tau_2 a_1 t}{t^{n-1} a_n}.$$

For $k_1 = 1, k_2 = 1, \tau_1 = 1, \tau_2 = 1$, Cor. 4 reduces to a result of Zargar [8, Cor.4].

2. Proofs of Theorems

Proof of Theorem 1: Suppose n is odd. Consider the polynomial

$$F(z)=(1-z^2)P(z)=(1-z^2)(a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0)$$
$$= -a_n z^{n+2} - a_{n-1}z^{n+1} + (a_n - a_{n-2})z^n + ...+ (a_2 - a_0)z + a_0$$
$$= -a_n z^{n+2} - a_{n-1}z^{n+1} + (k_1 a_n - a_{n-2})z^n - (k_1 - 1)a_n z^n + (k_2 a_{n-1} - a_{n-3})z^{n-1}$$
$$+ (k_2 - 1)a_{n-1}z^{n-1} + ... + (a_{2\lambda+2} - a_{2\lambda})z^{2\lambda+2} + (a_{2\lambda+1} - a_{2\lambda-1})z^{2\lambda+1}$$
$$+ (a_{2\lambda} - a_{2\lambda-2})z^{2\lambda} + ... + (a_3 - \tau_1 a_1)z^3 + (\tau_1 - 1)a_1 z^3 + (a_2 - \tau_2 a_0)z^2$$
$$+ (\tau_2 - 1)a_0 z^2 + a_1 z + a_0$$

For $|z|>1$, so that $\frac{1}{|z|^j}<1, \forall j = 1,2,......n$, we have , by using the hypothesis,

$$|F(z)| \ge |z|^n [|a_n z^2 + a_{n-1}z + (k_1 - 1)a_n| - \{k_1 a_n - a_{n-2} + \frac{k_2 a_{n-1} - a_{n-3}}{|z|} + \frac{(k_2 - 1)a_{n-1}}{|z|}$$
$$+ ......+ \frac{a_{2\lambda+2} - a_{2\lambda}}{|z|^{n-2\lambda-2}} + \frac{a_{2\lambda-1} - a_{2\lambda+1}}{|z|^{n-2\lambda-1}} + \frac{a_{2\lambda-2} - a_{2\lambda}}{|z|^{n-2\lambda}} + ...+ \frac{\tau_1 a_1 - a_3}{|z|^{n-3}}$$
$$+ \frac{(1-\tau_1)a_1}{|z|^{n-3}} + \frac{\tau_2 a_0 - a_2}{|z|^{n-2}} + \frac{(1-\tau_2)a_0}{|z|^{n-2}} + \frac{a_1}{|z|^{n-1}} + \frac{a_0}{|z|^n}\}]$$
$$> |z|^n [|a_n z^2 + a_{n-1}z + (k_1 - 1)a| - \{k_1 a_n - a_{n-2} + k_2 a_{n-1} - a_{n-3} + (k_2 - 1)a_{n-1}$$
$$+ ......+ a_{2\lambda+2} - a_{2\lambda} + a_{2\lambda+1} - a_{2\lambda-1} + a_{2\lambda} - a_{2\lambda-2} - a_{2\lambda} + ...+ a_3 - \tau_1 a_1$$
$$+ (1-\tau_1)a_1 + a_2 - \tau_2 a_0 + (1-\tau_2)a_0 + a_1 + a_0\}]$$
$$= |z|^n [|a_n z^2 + a_{n-1}z + (k_1 - 1)a_n| - \{k_1 a_n + (2k_2 - 1)a_{n-1} - 2(\tau_1 - 1)a_1$$

if

$$|a_n z^2 + a_{n-1}z + (k_1 - 1)a_n| > k_1 a_n + (2k_2 - 1)a_{n-1} - 2(\tau_1 - 1)a_1 - 2(\tau_2 - 1)a_0$$
$$- 2(\tau_2 - 1)a_0$$

or

$$\left|z^2 + \frac{a_{n-1}}{a_n} z + k_1 - 1\right| > \frac{1}{a_n}[k_1 a_n + (2k_2 - 1)a_{n-1} - 2(\tau_1 - 1)a_1 - 2(\tau_2 - 1)a_0]$$

This shows that all those zeros of F(z) whose modulus is greater than 1 lie in

$$\left|z^2 + \frac{a_{n-1}}{a_n} z + k_1 - 1\right| \le \frac{1}{a_n}[k_1 a_n + (2k_2 - 1)a_{n-1} - 2(\tau_1 - 1)a_1 - 2(\tau_2 - 1)a_0]$$

or

$$|z-\alpha||z-\beta| \le \frac{1}{a_n}[k_1 a_n + (2k_2 - 1)a_{n-1} - 2(\tau_1 - 1)a_1 - 2(\tau_2 - 1)a_0]$$

where α, β are the roots of the quadratic

$$z^2 + \frac{a_{n-1}}{a_n} z + k_1 - 1 = 0 .$$

Since the zeros of F(z) with modulus less than or equal to 1 already satisfy the above inequality, it follows that all the zeros of F(z)lie in

$$|z-\alpha||z-\beta| \le \frac{1}{a_n}[k_1 a_n + (2k_2 - 1)a_{n-1} - 2(\tau_1 - 1)a_1 - 2(\tau_2 - 1)a_0] .$$

Since the zeros of P(z) are also the zeros of F(z), , it follows that all the zeros of P(z)lie in

$$|z-\alpha||z-\beta| \le \frac{1}{a_n}[k_1 a_n + (2k_2 - 1)a_{n-1} - 2(\tau_1 - 1)a_1 - 2(\tau_2 - 1)a_0].$$

The case of even n follows similarly.

Proof of Theorem 2: Suppose n is odd. Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z^2)P(z) = (1-z^2)(a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0) \\
&= -a_n z^{n+2} - a_{n-1}z^{n+1} + (a_n - a_{n-2})z^n + ...+ (a_2 - a_0)z + a_0 \\
&= -a_n z^{n+2} - a_{n-1}z^{n+1} + (k_1 a_n - a_{n-2})z^n - (k_1 - 1)a_n z^n + (k_2 a_{n-1} - a_{n-3})z^{n-1} \\
&\quad + (k_2 - 1)a_{n-1}z^{n-1} + ... + (a_{2\lambda+2} - a_{2\lambda})z^{2\lambda+2} + (a_{2\lambda+1} - a_{2\lambda-1})z^{2\lambda+1} \\
&\quad + (a_{2\lambda} - a_{2\lambda-2})z^{2\lambda} + ... + (a_3 - \tau_1 a_1)z^3 + (\tau_1 - 1)a_1 z^3 + (a_2 - \tau_2 a_0)z^2 \\
&\quad + (\tau_2 - 1)a_0 z^2 + a_1 z + a_0
\end{aligned}$$

For $|z| > 1$, so that $\frac{1}{|z|^j} < 1, \forall j = 1,2,......n+1$, we have , by using the hypothesis,

$$\begin{aligned}
|F(z)| &\ge |z|^{n+1}[|a_n z + a_{n-1}| - \{\frac{k_1 a_n - a_{n-2}}{|z|} + \frac{(k_1 - 1)a_n}{|z|} + \frac{k_2 a_{n-1} - a_{n-3}}{|z|^2} + \frac{(k_2 - 1)a_{n-1}}{|z|^2} \\
&\quad + ...+ \frac{a_{2\lambda+2} - a_{2\lambda}}{|z|^{n-2\lambda-1}} + \frac{a_{2\lambda-1} - a_{2\lambda+1}}{|z|^{n-2\lambda}} + \frac{a_{2\lambda-2} - a_{2\lambda}}{|z|^{n-2\lambda+1}} + ...+ \frac{\tau_1 a_1 - a_3}{|z|^{n-2}} \\
&\quad + \frac{(\tau_1 - 1)a_1}{|z|^{n-2}} + \frac{\tau_2 a_0 - a_2}{|z|^{n-1}} + \frac{(\tau_2 - 1)a_0}{|z|^{n-1}} + \frac{a_1}{|z|^n} + \frac{a_0}{|z|^{n+1}}\}] \\
&> |z|^{n+1}[|a_n z + a_{n-1}| - \{k_1 a_n - a_{n-2} + (k_1 - 1)a_n + k_2 a_{n-1} - a_{n-3} + (k_2 - 1)a_{n-1} \\
&\quad + ...+ a_{2\lambda+2} - a_{2\lambda} + a_{2\lambda-1} - a_{2\lambda+1} + a_{2\lambda-2} - a_{2\lambda} + ...+ \tau_1 a_1 - a_3 \\
&\quad + (\tau_1 - 1)a_1 + \tau_2 a_0 - a_2 + (\tau_2 - 1)a_0 + a_1 + a_0\}] \\
&= |z|^{n+1}[|a_n z + a_{n-1}| - \{(2k_1 - 1)a_n + (2k_2 - 1)a_{n-1} - 2a_{2\lambda} - 2a_{2\lambda+1} \\
&\quad + 2\tau_1 a_1 + 2\tau_2 a_0\}] \\
&> 0
\end{aligned}$$

if

$$|a_n z + a_{n-1}| > (2k_1 - 1)a_n + (2k_2 - 1)a_{n-1} - 2a_{2\lambda} - 2a_{2\lambda+1} + 2\tau_1 a_1 + 2\tau_2 a_0$$

i.e.

$$\left|z + \frac{a_{n-1}}{a_n}\right| > \frac{(2k_1 - 1)a_n + (2k_2 - 1)a_{n-1} - 2a_{2\lambda} - 2a_{2\lambda+1} + 2\tau_1 a_1 + 2\tau_2 a_0}{a_n}$$

This shows that all those zeros of F(z) whose modulus is greater than 1 lie in

$$\left|z + \frac{a_{n-1}}{a_n}\right| \le \frac{(2k_1 - 1)a_n + (2k_2 - 1)a_{n-1} - 2a_{2\lambda} - 2a_{2\lambda+1} + 2\tau_1 a_1 + 2\tau_2 a_0}{a_n}.$$

Since the zeros of F(z) with modulus less than or equal to 1 already satisfy the above inequality, it follows that all the zeros of F(z) lie in the disk

$$\left|z + \frac{a_{n-1}}{a_n}\right| \le \frac{(2k_1 - 1)a_n + (2k_2 - 1)a_{n-1} - 2a_{2\lambda} - 2a_{2\lambda+1} + 2\tau_1 a_1 + 2\tau_2 a_0}{a_n}.$$

Since the zeros of P(z) are also the zeros of F(z), , it follows that all the zeros of P(z)lie in the disk

$$\left| z + \frac{a_{n-1}}{a_n} \right| \le \frac{(2k_1 - 1)a_n + (2k_2 - 1)a_{n-1} - 2a_{2\lambda} - 2a_{2\lambda+1} + 2\tau_1 a_1 + 2\tau_2 a_0}{a_n} .$$

The case of even n follows similarly.

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