International Journal of Engineering Research and Development
e-ISSN: 2278-067X, p-ISSN: 2278-800X, www.ijerd.com
Volume 10, Issue 5 (May 2014), PP.21-30

Wavelet– Galerkin Solution for Boundary Value Problems

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Abstract:- Wavelet-Galerkin technique has very important advantages over classical finite difference and finite element method. In this paper, we have made an attempt to develop a technique for Wavelet-Galerkin solution of Neumann and mixed boundary value problem in one dimension in parallel to the work of J. Besora [5]. The Taylor's approach has been used to include Neumann condition in Wavelet-Galerkin setup. The test examples are given to validate techniques with exact solutions.

Keywords:- Boundary Value Problems, Wavelets, Scaling Function, Connection Coefficients, Wavelet Coefficient ,Wavelet- Galerkin Method.

I. INTRODUCTION

Wavelet is an important area of mathematics and now a days it becomes an important tool for applications in many areas of science and engineering.

The orthonormal bases of compactly supported wavelets for the space of square-integral function $L^2(\mathbb{R})$ was constructed by Daubecheis in 1988 (see [7]). Those wavelets are bases in a Galerkin method to solve Dirichlet as well as Neumann mixed boundary value problem. The approximations of partial differential equations with wavelet basis are more attentive, because the reality of orthogonality of compactly supported wavelets. The concepts of Multiresolution Analysis based Fast wavelet transform algorithm have given great momentum to make wavelet best approximations of ODE's and PDE's. Wavelet-Galerkin technique is frequently used now a days, and its numerical solution of partial differential equations have been developed by several researchers.

Many contributions have been done in the area of wavelets and differential equations by authors like Beylkin et al.[4], Jawerth et al. [9], Qian et al.[14], Qian et al. [15], Williams et al. [16], Williams et al. [19]. Several researchers now a day's uses Daubechies wavelets as bases in a Galerkin method to solve boundary value problem. The contribution in this area is due to the remarkable work by Latto et al. [10], Xu et al. [21], [22], Williams et al. [16], [17] , [18], Mishra et al [13], Jordi Besora [5] and Amartunga et al. [1], [2], [3]. It is known that the problems with periodic boundary conditions have been handled successfully. Fictious boundary approach with Dirichlet boundary conditions have been applied by Dianfeng et al. [8] in analyzing SH wave equation. Latto et al. [10] presents a connection coefficients for j=0 and N=6. The problem with Distinct boundary conditions is developed by Jordi Besora [5].

In this paper, we have proposed wavelet-Galerkin approximation to the Neumann and mixed boundary value problem, using compactly supported wavelets as basis functions introduced by Daubechies [6], [7]. We have used Taylors approach to deal with Neumann and mixed Boundary conditions.

II. WAVELETS

An oscillatory function $\psi(x) \in L^2(\mathbb{R})$ is a wavelet if it has following properties:

- $\psi(x)$ is n times differentiable and its derivatives are continuous.
- $\psi(x)$is well localized both in time and frequency domains, i.e. $\psi(x)$ and its derivatives must decay very rapidly. For frequency localization $\hat{\psi}(\omega)$ must decay sufficiently fast as $|\omega| \longrightarrow \infty$ and that $\hat{\psi}(\omega)$ becomes flat in the neighbourhood of $\omega = 0$. The flatness is associated with number of vanishing moments of $\psi(x)$, i.e.

$$\int_{-\infty}^{\infty} x^k \psi(x) dx = 0$$

or equivalently

$$\frac{d^k}{d\omega^k}\hat{\psi}(\omega) = 0$$

for $k = 0,1, \dots n$.

- in the sense that the larger number of vanishing moments more is the flatness when ω is small.

$$\int_{-\infty}^{\infty} \frac{|\psi(\omega)|^2}{|\omega|} d\omega < \infty$$

Daubechies wavelets are compactly supported functions. Since they have non zero values within a finite interval and zero values elsewhere in some intervals and they can be used for representation of solution of BVP. In order to have multiresolution properties, the scaling function is defined as $\phi(x)$, and it must satisfy,

$$\phi(x) = \sum_{k=0}^{L-1} a_k \phi(2x - k) \tag{1}$$

or

$\phi(x) = \sqrt{2}\sum_{k=0}^{N-1} C_k \phi(2x - k)$; when $a_k = \sqrt{2}C_k$ with the property that $\sum_{k=0}^{N-1} a_k = \sqrt{2}$.
where L denotes the genus of the Daubechies wavelet. The functions generated with these coefficients will have supp $\phi = [0, L-1]$ and $(\frac{L}{2} - 1)$ vanishing wavelet moments.

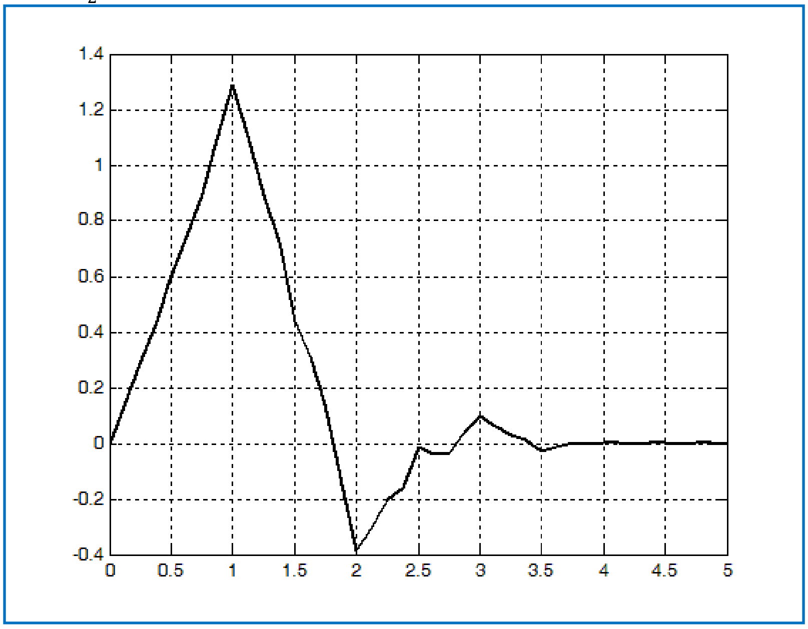

Fig.1: Daubechies Scaling function with L=6 and j=0

The basic wavelet is defined in terms of scaling function by,

$$\psi(x) = \sum_{k=-L}^{1} (-1)^k a_{1-k} \phi_k(2x - k) \tag{2}$$

A family of scaling functions is generated by translation from

$$\phi_k(x) = \phi(x - k) \tag{3}$$

Similar a family of wavelets is denoted by translation and scaling from

$$\psi_{j,k}(x) = \psi(2^j x - k) \tag{4}$$

Since $\int \phi(x)dx = 1$; the scaling function satisfies the three conditions.

1.
$$\sum_{k=0}^{L-1} a_k = 2 \tag{5}$$
2. The orthonormality condition of the scaling function
$$\int \phi(x-k)\phi(x-m)dx = \delta_{k,m}$$
Implies that
$$\sum_{k=0}^{L-1} a_k a_{k-2m} = \delta_{0,m} \tag{6}$$

where δ is a kroneker delta function.

3. The moment of the smooth wavelet function to be zero.

$$\int x^m \psi(x)dx = 0$$

implies that

$$\sum_{k=0}^{L-1}(-1)^k k^m a_k = 0 \tag{7}$$

for $m = 0,1, \dots \frac{L}{2} - 1$

The relation between scaling function and wavelet function can be expressed as ;

$$V_{j+1} = V_j \oplus W_j$$

It implies that, for any integer m,

$$\int \phi(x)\psi(x-m)dx = 0 \tag{8}$$

Where $\oplus$ denotes the direct sum and V_j and W_j be the subspaces generated, respectively as the L^2- closure of the linear spans of

$$\phi_{j,k}(x) = 2^j \phi(2^j x - k) \text{ and}$$
$$\psi_{j,k}(x) = 2^j \psi(2^j x - k), \qquad k \in \mathbb{Z}$$

Using condition (8) we can write

$$V_0 \subset V_1 \subset \cdots \subset V_j \subset V_{j+1}$$

And

$$V_{j+1} = V_0 \oplus W_0 \oplus W_1 \oplus W_2 \oplus \cdots \oplus W_j$$

where j is the dilation parameter as scale. The support of the scale function $\phi(2^j x - k)$ for a certain value of j and L is given as

$$supp\left(\phi(2^j x - k)\right) = \left[\frac{k}{2^j}, \frac{L+k-1}{2^j}\right]$$

Each function $f(x) \in L^2(\mathbb{R})$, can be written in the form

$$f(x) = \sum_k 2^j c_k \phi(2^j x - k) \tag{9}$$

and this should be satisfied the convergence property

$$\left\| f - \sum_k c_k \phi(2^j x - k) \right\| \le 2^{-jp} C \|f^p\|$$

where

$$c_k = \int f(x)\phi(2^j x - k)\, dx$$

and C and p are constants.

III. MULTIRESOLUTION ANALYSIS

Construction of smooth wavelets with compact support is a major task in todays research scenario. Meyer[6] and Mallat [11] found conditions which we called multiresolution analysis which is use by Daubchies to construct wavelet of compact support having arbitrary smoothness.

A MRA of $L^2(\mathbb{R})$ is defined as a sequence of closed subspace of $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ and a scaling function ϕ satisfy the following conditions.

1. The space V_j are nested i.e.
$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots,$$
2. The space $L^2(\mathbb{R})$ is a closure of the union of all V_j and the intersection of all V_j is empty. i.e.
$\overline{\cup_j V_j} = L^2(\mathbb{R})$ and $\cap_j V_j = \{0\}$
3. $f(x) \in V_j \iff f(2x) \in V_{j+1}$; $\forall j \in \mathbb{Z}$
4. $f(x) \in V_0 \iff f(x-k) \in V_0$; $\forall k \in \mathbb{Z}$
5. $\{\phi(x-k)\}$ is an orthonormal basis for the space V_0.

IV. WAVELET-GALERKIN METHOD

Galerkin Method was introduced by V.I Galerkin. We can discuss this using a one-dimensional differential equation.

$$Lu(x) = f(x); \qquad 0 \le x \le 1; \tag{10}$$

where

$$L = -\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) + b(x)u(x)$$

with boundary conditions, $u(0) = 0$ and $u(1) = 0$.

Where a, b and f are given real valued continuous function on $[0,1]$. We also assume that L is a elliptic differential operator.
Consider, $\{v_j\}$ is a complete orthonormal basis of $L^2([0,1])$ and every v_j $C^2 \in ([0,1])$ such that,

$$v_j(0) = 0 \qquad v_j(1) = 0$$

We can select the finite set Λ of indices j and then consider the subspace S,

$$S = span\{v_j;\ j \in \Lambda\}.$$

Approximate solution u_s can be written in the form,

$$u_s = \sum_{k\in\Lambda} x_k v_k \in S \tag{11}$$

where each x_k is scalar. We may determine x_k by seeing the behaviour of u_s as it look like a true solution on S. i.e.

$$\langle L\, u_s , v_j \rangle = \langle f, v_j \rangle \qquad \forall\, j \in \Lambda, \tag{12}$$

such that the boundary conditions $u_s(0) = 0$ and $u_s(1) = 0$ are satisfied. Substituting u_s values into the equation (12),

$$\sum_{k\in\Lambda} \langle L\, v_k , v_j \rangle\, x_k = \langle f, v_j \rangle \qquad \forall\, j \in \Lambda \tag{13}$$

Then this equation can be reduced in to the linear system of equation of the form

$$\sum a_{jk}\, x_k = y_i \tag{14}$$

or

$$AX=Y$$

where $A = \left[a_{jk}\right]_{j,k\in\Lambda}$ and $a_{jk} = \langle L\, v_k , v_j \rangle$, x denotes the vector $\{x_k\}_{k\in\Lambda}$ and y denotes the vector $\{y_k\}_{k\in\Lambda}$. In the Galerkin method, for each subset Λ, we obtain an approximation $u_s \in$ S by solving linear system (14).
If u_s converges to u then we can find the actual solution.

Our main concern is the method of linear system (14) by choosing a wavelet Galerkin method. The matrix A should have a small condition number to obtain stability of solution and A should sparse to perform calculation fast.
Similarly we can do the same thing in above set up.
Let $\psi_{j,k}(x) = 2^j \psi(2^j x - k)$ (15)
is a basis for $L^2([0,1])$ with boundary conditions

$$\psi_{j,k}(0) = \psi_{j,k}(1) = 0 \qquad \forall\, j,k \in \Lambda \text{ and } \psi_{j,k} \text{ is } C^2.$$

We can replace equations (12) and (13) by

$$u_s = \sum_{j,k\in\Lambda} x_{j,k} \psi_{j,k}$$

and

$$\sum_{j,k\in\Lambda} \langle L\, \psi_{j,k} , \psi_{l,m} \rangle\, x_{j,k} = \langle f, \psi_{l,m} \rangle \qquad \forall\, l,m \in \Lambda$$

So that $AX=Y$.
Where $A = \left[a_{l,m;j,k}\right]_{(l,m),(j,k)\in\Lambda}$; $\quad X = \left(x_{j,k}\right)_{(j,k)\in\Lambda} \quad Y = \left(y_{l,m}\right)_{(l,m)\in\Lambda}$
Then

$$a_{l,m;j,k} = \langle L\, \psi_{j,k} , \psi_{l,m} \rangle$$

$$y_{l,m} = \langle f, \psi_{l,m} \rangle$$

Where l, m and j, k represent respectively row and column of A.
This is an accurate method to find the solution of partial differential equation.

V. CONNECTION COEFFICIENT

To find the solution of differential equation using the Wavelet Galerkin technique we have to find the connection coefficients which is also explored in Latto et al.([10]),

$$\Omega^{d_1 d_2}_{\ell_1 \ell_2} = \int_{-\infty}^{\infty} \Phi^{d_1}_{\ell_1}(x)\, \Phi^{d_2}_{\ell_2}(x) dx \tag{16}$$

Taking derivatives of the scaling function d times, we get

$$\phi^d(x) = 2^d \sum_{k=0}^{L-1} a_k \phi^d_k(2x-k) \tag{17}$$

We can simplify equation (16) then for all $\Omega^{d_1 d_2}_{\ell_1 \ell_2}$ gives a system of linear equation with unknown vector $\Omega^{d_1 d_2}$

$$T\Omega^{d_1 d_2} = \frac{1}{2^{d-1}} \Omega^{d_1 d_2} \tag{18}$$

where $d = d_1 + d_2$ and $T = \sum_i a_i\, a_{q-2l+i}$. These are so called scaling equation.
But this is the homogeneous equation and does not have a unique nonzero solution. In order to make the system inhomogeneous, one equation is added and it derived from the moment equation of the scaling function ϕ. This is the normalization equation,

$$d! = (-1)^d \sum_l M^d_l\, \Omega^{0,d}_l$$

Connection coefficient $\Omega^{0,d}_l$ can be obtained very easily using $\Omega^{d_1 d_2}_l$,

$$\Omega_l^{0,d} = \int \phi^{d_1} \phi_l^{d_2} dx$$
$$= [\phi^{d_1-1} \phi_l^{d_2}]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \phi^{d_1-1} \phi^{d_2+1} dx$$

As a result of compact support wavelet basis functions exhibit, the above equation becomes

$$\Omega^{d_1 d_2} = -\int_{-\infty}^{\infty} \phi^{d_1-1} \phi_l^{d_2+1} dx \qquad (19)$$

After d_1 integration ,

$\Omega_l^{d_1 d_2} = (-1)^d \int_{-\infty}^{\infty} \phi^{d_1-1} \phi_l^{d_2+d_3} dx = (-1)^d \, \Omega_l^{0,d}$

The moments M_i^k of ϕ_i are defined as

$$M_i^k = \int_{-\infty}^{\infty} x^k \phi_i(x) dx$$

With $M_0^0 = 1$

Latto et al derives a formula as

$$M_i^m = \frac{1}{2(2^m-1)} \sum_{t=0}^{m} \binom{m}{t} i^{m-t} \sum_{l=0}^{t-1} \binom{t}{l} \sum_{i=0}^{L-1} a_i i^{t-l} \qquad (20)$$

Where a_i's are the Daubechies wavelet coefficients. Finally, the system will be

$$\begin{pmatrix} T - \frac{1}{2^{d-1}} I \\ M^d \end{pmatrix} \quad \Omega^{d_1 d_1} = \begin{pmatrix} 0 \\ d! \end{pmatrix} \qquad (21)$$

Matlab software is used to compute the connection coefficient and moments at different scales. Latto et al [10] computed the coefficients at j=0 and L=6 only. The computation of connection coefficients at different scales have been done by using the program given in Jordi Besora [5]. The scaling function at *j*=0 and *L*=6 connection coefficient prepared by Latto et al [10] is given in Table 1 and Table 2 respectively.

Table I: Scaling function at j=0 and L=6 have been provided by Latto et al. [10]

x	Phi(x)	x	Phi(x)
0.000	0	**2.500**	-0.014970591
0.125	0.133949835	**2.625**	-0.03693836
0.250	0.284716624	**2.750**	-0.040567571
0.375	0.422532739	**2.875**	0.037620632
0.500	0.605178468	**3.000**	0.095267546
0.625	0.743571274	**3.125**	0.062104053
0.750	0.89811305	**3.250**	0.02994406
0.875	1.090444005	**3.375**	0.011276602
1.000	1.286335069	**3.500**	-0.031541303
1.125	1.105172581	**3.625**	-0.013425276
1.250	0.889916048	**3.750**	0.003025131
1.375	0.724108826	**3.875**	-0.002388515
1.500	0.441122481	**4.000**	0.004234346
1.625	0.30687191	**4.125**	0.001684683
1.750	0.139418882	**4.250**	-0.001596798
1.875	-0.125676646	**4.375**	0.000149435
2.000	-0.385836961	**4.500**	0.000210945
2.125	-0.302911152	**4.625**	-7.95485E-05
2.250	-0.202979935	**4.750**	1.05087E-05
2.375	-0.158067602	**4.875**	5.23519E-07
		5.000	-3.16007E-20

Table II: Connection coefficients at *j*=0 and *L*=6 have been provided by Latto et al. [13] using $\Omega[n-k=\phi''x-k\phi(x-n)dx$

$\Omega[-3]$	5.357142857141622e-003
$\Omega[-2]$	1.142857142857171e-001
$\Omega[-1]$	-8.761904761905105e-001
$\Omega[0]$	3.390476190476278e+000
$\Omega[1]$	-5.267857142857178e+000
$\Omega[2]$	3.390476190476152e+000
$\Omega[2]$	-8.761904761904543e-001
$\Omega[3]$	1.142857142857135e-001
$\Omega[4]$	5.357142857144167e-003

VI. TEST PROBLEMS

Consider

$$\frac{d^2u(x)}{dx^2}+\beta u(x)=f \tag{22}$$

Now we use Wavelet-Galerkin method solution
Here, we consider $L=6$ *and* $j=0$
We can write the solution of the differential equation (22) is,

$$u(x)=\sum_{k=L-1}^{2^j} c_k 2^{\frac{j}{2}}\Phi(2^j x-k), \qquad x\in[0,1]$$

$$=\sum_{k=-5}^{1} c_k\Phi(x-k), \qquad x\in[0,1] \tag{23}$$

where c_k are the unknown constant co-efficient
Substitute (23) in (22) we get

$$\frac{d^2}{dx^2}\sum_{k=-5}^{1} c_k\Phi(x-k)+\beta\sum_{k=-5}^{1} c_k\Phi(x-k)=0$$

$$\sum_{k=-5}^{1} c_k\phi''(x-k)+\beta\sum_{k=-5}^{1} c_k\phi(x-k)=0$$

Taking inner product with $\phi(x-k)$
We have

$$\sum_{k=-5}^{1} c_k\int_{\frac{1-L}{2^j}}^{\frac{L-1+2^j}{2^j}}\phi''(x-k)\phi(x-n)\,dx+\beta\sum_{k=-5}^{1} c_k\int_{\frac{1-L}{2^j}}^{\frac{L-1+2^j}{2^j}}\phi(x-k)\phi(x-n)\,dx=0$$

$$\Rightarrow\sum_{k=-5}^{1} c_k\Omega[n-k]+\beta\sum_{k=-5}^{1} c_k\delta_{n,k}=0 \tag{24}$$

$$n=1-L,2-L,\ldots 2^j$$

i.e; $n=-5,-4,\ldots,0\ 1$
where

$$\Omega[n-k]=\int\phi''(x-k)\phi(x-n)dx$$

$$\delta_{n,k}=\int\phi(x-k)\phi(x-n)dx$$

By using Neumann Boundary conditions
$u(0)=1;\quad u'(1)=0$

Considering left and right boundary conditions we can write

$$u(0)=\sum_{k=-5}^{1} c_k\phi(-k)=1 \tag{25}$$

$$u'(1)=\sum_{k=-5}^{1} c_k\phi(1-k)=0 \tag{26}$$

We use Taylors method to approximate derivative on the right side

$$u(x) = u(a) + hu'(a) + \frac{h^2}{2!}\,u''(a) + \cdots\cdots$$

which gives the approximation of derivative at the boundary using forward, backward or central differences. Equation (25) and (26) represent the relation of the coefficient c_k.

We can replace first and last equations of (24) using (25) and (26) respectively. Then we can get the following matrix with $L=6$

$$TC=B$$

$$T = \begin{bmatrix} 0 & \phi(4) & \phi(3) & \phi(2) & \phi(1) & 0 & 0 \\ \Omega[1] & \Omega[0] & \Omega[-1] & \Omega[-2] & \Omega[-3] & \Omega[-4] & \Omega[-5] \\ \Omega[2] & \Omega[1] & \Omega[0] & \Omega[-1] & \Omega[-2] & \Omega[-3] & \Omega[-4] \\ \Omega[3] & \Omega[2] & \Omega[1] & \Omega[0] & \Omega[-1] & \Omega[-2] & \Omega[-3] \\ \Omega[4] & \Omega[3] & \Omega[2] & \Omega[1] & \Omega[0] & \Omega[-1] & \Omega[-2] \\ \Omega[5] & \Omega[4] & \Omega[3] & \Omega[2] & \Omega[1] & \Omega[0] & \Omega[-1] \\ 0 & 0 & p(1) & p(2) & p(3) & p(4) & 0 \end{bmatrix}$$

$$p(1) = \phi(4) - \phi(4-h)$$
$$p(2) = \phi(3) - \phi(3-h)$$
$$p(3) = \phi(2) - \phi(2-h)$$
$$p(4) = \phi(1) - \phi(1-h)$$

$$C = \begin{bmatrix} c_{-5} \\ c_{-4} \\ c_{-3} \\ c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

1. Suppose given Boundary Value Problem is,

$$u_{xx} = -2 \tag{27}$$

with boundary conditions $u(0) = 1$ and $u'(1) = 0$

The exact solution is,

$$u(x) = -x^2 + 2x + 1$$

c_{-5} =-0.997475258266510

c_{-4} = -0.877910629139006

c_{-3} =0.127857696605495

c_{-2} =1.054541418880270

c_{-1} =1.087386803133082

c_0 = 0.247680298288321

c_1 = -0.506012992523331

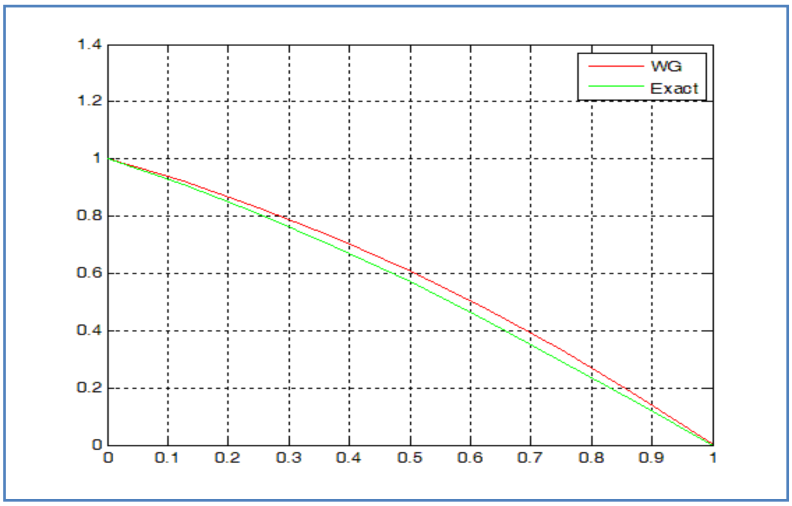

Fig.2: Wavelet-Galerkin Solution for $u_{xx} = -2,\ u(0) = 1$ and $u'(1) = 0$

2. The given boundary value Problem is,

$$u'' + u = 0; \qquad u(0) = 1 \ \ and \ \ u'(1) = 0$$

The exact solution is,

$$u(x) = \cos x + \tan 1 \sin x$$

c_{-5} =-16.784099470868910

c_{-4} =-15.622648376150911

c_{-3} =-8.883687788040305

c_{-2} = -3.026098858023991

c_{-1} = 0.579267070363675

c_0 = 1.960144587700136

c_1 = 1.418474146775382

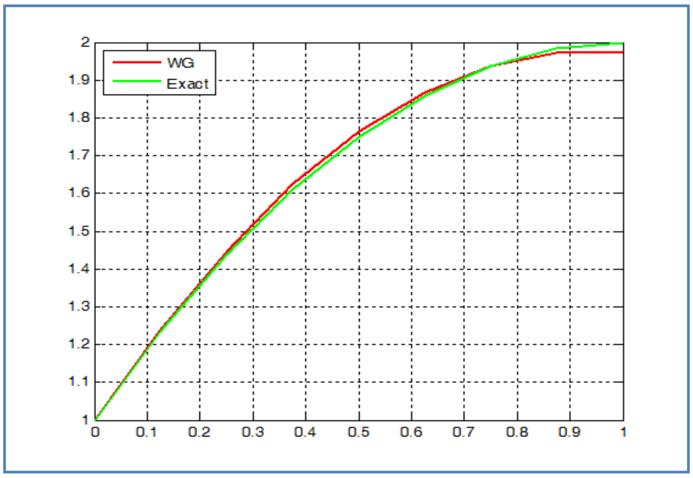

Fig.3: Wavelet-Galerkin Solution for $u'' + u = 0; \quad u(0) = 1$ and $u'(1) = 0$

3. The given boundary value Problem is,

$$u'' + u = 0; \qquad u(0) = 1 \ \ and \ \ u'(1) + u(1) = 0$$

The exact solution is,

$$u(x) = \cos x + (\tan 2 - \sec 2) \sin x$$

c_{-5} =-13.541294401791857

c_{-4} =-12.126299651408310

c_{-3} = -6.346329455749815

c_{-2} = -1.657781810857689

c_{-1} =0.790083941059115

$c_{0=}$ 1.133626147467154

$c_{1=}$ 0.186034129844783

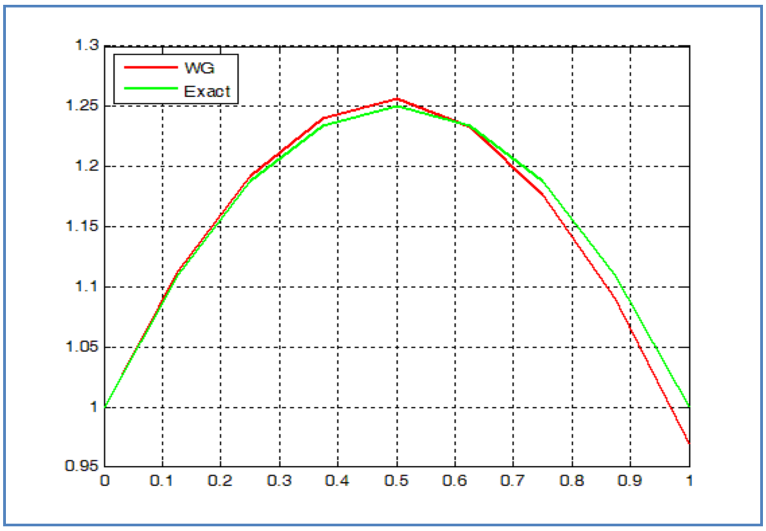

Fig.4: Wavelet-Galerkin Solution for $u'' + u = 0$; $u(0) = 1$ and $u'(1) + u(1) = 0$

VII. CONCLUSIONS

Wavelet method has shown a very powerful numerical technique for the stable and accurate solution of given boundary value problem. The comparison shows that the exact solution correlates with numerical solution, using Daubechies wavelets.

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